

DUAL SPACE OF THE SPACE OF POLYNOMIAL FUNCTIONS OF BOUNDED DEGREE ON AN EMBEDDED MANIFOLD

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ABSTRACT. Let $O(U)$ denote the algebra of holomorphic functions on an open subset $U \subset \mathbb{C}^n$ and $Z \subset O(U)$ its finite-dimensional vector subspace. By the theory of least space of de Boor and Ron, there exists a projection T_b from the local ring $O_{n,b}$ onto the space Z_b of germs of elements of Z at b . At general $b \in U$ its kernel is an ideal and induces a structure of an Artinian algebra on Z_b . In particular, it holds at points where k -th jets of elements of Z form a vector bundle for each $k \leq \dim_{\mathbb{C}} Z_b - 1$. Using T_b we define the Taylor projector of order d on an embedded curve $X \subset \mathbb{C}^m$ at a general point $a \in X$, generalising results of Bos and Calvi. It is a retraction of $O_{X,a}$ onto the set of the polynomial functions on X_a of degree up to d . For an embedded manifold $X \subset \mathbb{C}^m$, we introduce a set of higher order tangents following Bos and Calvi and show a zero-estimate for a system of generators of the maximal ideal of $\mathbb{C}\{t - b\}$ at general $b \in X$. It means that X is embedded in \mathbb{C}^n in not very highly transcendental manner at a general point.

Keywords: • Taylor projector • one point interpolation on manifolds • least operator • higher order tangents • zero-estimate • transcendency index

MSC (2000): primary 32B15, secondary 11J82, 13J07, 32E30, 41A10, 41A63

1. INTRODUCTION

The motivation of this paper is applications of least spaces of de Boor and Ron and generalisation of the theory of Bos and Calvi on Taylor projector on a plane algebraic curve. In particular, the main problem is to clarify the nature of “singularities of an affine embedding of a manifold” found by Bos and Calvi in the plane curve case.

First we introduce the least operator of de Boor and Ron. It is a simple and useful way to construct a dual space of a vector space of holomorphic functions. Let U be an open subset of an affine space \mathbb{C}^n and let $O_n(U)$ denote the ring of holomorphic functions on U . Take $f(t) \in O_n(U)$ ($t := (t_1, \dots, t_n)$) and $b := (b_1, \dots, b_n) \in U$. The *least part* $f_b \downarrow$ of f at b means the non-zero homogeneous part of the lowest degree of the power series expansion of f with respect to the coordinates $t' := t - b$ centred at b . Replacing the variable by the corresponding Greek letter, $f_b \downarrow$ may be considered as an element of $\mathbb{C}[\tau]$. For example

$$((t_1 - b_1)^p(t_2 - b_2)^q + (t_1 - b_1)^{p+1}(t_2 - b_2)^q)_b \downarrow = \tau_1^p \tau_2^q \in \mathbb{C}[\tau].$$

A vector subspace $W \subset \mathbb{C}[\tau]$ is defined to be *D-invariant* if it is closed with respect to the partial differentiations with respect to τ_1, \dots, τ_n . If Z is a vector subspace of $O_{n,b}$ or $O(U)$, then $Z_b \downarrow \subset \mathbb{C}[\tau]$ denotes the linear span $\text{Span}(f_b \downarrow : f \in Z)$ over \mathbb{C} of least parts of elements of W . The

This work is partly supported by JSPS KAKENHI (21540054) and partly done during the author's stay at Sydney University with the aid of JSPS Bilateral Joint Projects/Seminars (2010).

Date: January 12, 2013.

mapping

$$\downarrow: Z \longrightarrow Z_b \downarrow, \quad f \longmapsto f_b \downarrow$$

is called the *least operator*. Let $S_{n,b} : \mathbb{C}[\tau] \times \mathcal{O}_{n,b} \longrightarrow \mathbb{C}$ denote the ordinary sesquilinear form (see §5). The restriction $S_Z : Z_b \downarrow \times Z \longrightarrow \mathbb{C}[\tau]$ of $S_{n,b}$ is found to be non-degenerate by de Boer and Ron [BR₁], [BR₂]. Let us define the projector $T_{Z,b} : \mathcal{O}_{n,b} \longrightarrow Z_b$ as the adjoint linear mapping of the inclusion $Z_b \downarrow \longrightarrow \mathcal{O}_{n,b} \downarrow = \mathbb{C}[\tau]$. Let U_Z^{inv} denote the set of D -invariant points of Z in U and U_Z^{bdl} the set of points of U where all the k -th jet of elements of Z form a vector bundle in a small neighbourhood, for each $k \leq \dim_{\mathbb{C}} Z - 1$. We prove the following.

Corollary 3.7, Theorem 4.2. *The set U_Z^{bdl} is invariant under biholomorphic transformation of U . It is non-empty and analytically open and $U_Z^{\text{bdl}} \subset U_Z^{\text{inv}}$.*

This is the key theorem of this paper but it can be proved simply describing $\mathbb{C}[\Phi]_b^d \downarrow$ by the language of “formal theory of differential equations”. The author is suggested this method by Tohru Morimoto.

Theorem 8.1. *Let U be an open subset of \mathbb{C}^n and Z a finite dimensional vector subspace of $\mathcal{O}_n(U)$. Then U_Z^{inv} is invariant under biholomorphic transformation of U and the vector space Z_b has a structure of an Artinian factor algebra of $\mathcal{O}_{n,b}$ through the projector $T_{Z,b} : \mathcal{O}_{n,b} \longrightarrow Z_b$ at each $b \in U_Z^{\text{inv}}$. This structure is unique up to canonical isomorphism.*

The result above traces back to the original theory of Bos and Calvi [BC₁]. Let X be a complex manifold of an open subset $U \subset \mathbb{C}^m$ and $\mathcal{O}_{X,a}$ the local algebra of the germs of holomorphic functions on X at a . The vector space of polynomials in \mathbf{x} of degrees at most d is denoted by $\mathbb{C}[\mathbf{x}]^d \subset \mathbb{C}[\mathbf{x}]$. We put $P^d(X_a) := \mathbb{C}[\mathbf{x}]^d|_{X_a} \subset \mathcal{O}_{X,a}$, the vector subspace of polynomial function of degree at most d . Let

$$\Phi := (\Phi_1, \dots, \Phi_m) : \mathbb{C}_b^n \longrightarrow \mathbb{C}_a^m$$

be a local parametrisation of X , which means the germ of a left inverse of a local chart of manifold X at a . In this paper, we express an analytic mapping germ by the upper case of a bold Greek letter and the algebra homomorphism induced by it is denoted by the lower case of the corresponding letter as

$$\varphi : \mathcal{O}_{m,a} \longrightarrow \mathcal{O}_{n,b}, \quad f \longmapsto f \circ \Phi.$$

Let

$$\mathbb{C}[\Phi]^d := \varphi(\mathbb{C}[\mathbf{x}]^d) \subset \mathcal{O}_{n,b}$$

denote the vector subspace of all the pull-backs of elements of $\mathbb{C}[\mathbf{x}]^d$ by Φ . Following Bos and Calvi, we introduce a special set $D_{X,a}^{\varphi,d}$ of higher order tangents of X at a as the set of push-forwards of elements of $\mathbb{C}[\Phi]_b^d \downarrow$ by Φ , where $\mathbb{C}[\Phi]_b^d \downarrow$ is considered as a set of higher order tangents of \mathbb{C}^n at b . It can be expressed as $D_{X,a}^{\varphi,d} := {}^s\varphi(\mathbb{C}[\Phi]_b^d) \subset \mathbb{C}[\mathbf{x}]$, using the adjoint homomorphism ${}^s\varphi : \mathbb{C}[\tau] \longrightarrow \mathbb{C}[\mathbf{x}]$ of φ . It is a dual space of $P^d(X_a)$ with respect to the sesquilinear form induced by $S_{n,b}$ and we call its elements *Bos-Calvi tangents*. We define the Taylor projector

$$T_a^{\varphi,d} : \mathcal{O}_{X,a} \longrightarrow P^d(X_a)$$

of degree d on $X \subset \mathbb{C}^m$ at a as the adjoint homomorphism of the inclusion $D_{X,a}^{\varphi,d} \longrightarrow \mathbb{C}[\tau]$. These $T_a^{\varphi,d}$ and $D_{X,a}^{\varphi,d}$ are defined using a local parametrisation φ of X around a .

In the case of a plane algebraic curve, Bos and Calvi [BC₂] prove that $T_a^{\varphi,d}$ is independent of the choice of φ if and only if the kernel of this projector is an ideal and that this condition actually holds at all but finite points on X , using the Wronskian. Bos and Calvi [BC₂] give the name “Taylorian property” to the independence of $T_a^{\varphi,d}$ from the local parametrisation φ . This property is equivalent to the condition that the powers of monomials appearing in $\mathbb{C}[\Phi]_b^d \subset \mathbb{C}[\tau]$ form a *gap-free* sequence. In this paper we generalise the theorem of Bos and Calvi to analytic curves with larger codimensions as follows.

Theorem 12.1. *Let X be a 1-dimensional regular complex submanifold of an open subset of \mathbb{C}^m . Then, for any $d \in \mathbb{N}$, the following three properties on $a \in X$ are equivalent.*

- (1) *Point a is a bundle point of all the jet spaces of $\mathbb{C}[\Phi]_b^d$.*
- (2) *The powers of monomials appearing in $\mathbb{C}[\Phi]_b^d \subset \mathbb{C}[\tau]$ form a gap-free sequence i.e. a is D -invariant of degree d*
- (3) *Point a is Taylorian of degree d .*

The D -invariance property is nothing but the gap-free property above but for a higher-dimensional submanifold, D -invariance does not mean the Taylorian property and the set $D_{X,a}^{\varphi,d}$ of higher order tangents depends upon the local parametrisation Φ . Still we have the following.

Remark 11.6. *Let X be a analytic submanifold of an open subset of \mathbb{C}^m . For any point $a \in X$, the following conditions are equivalent for each local parametrisation φ .*

- (1) *The set $D_{X,a}^{\varphi,d}$ of Bos-Calvi tangents is D -invariant at a .*
- (2) *The space of annihilators $(D_{X,a}^{\varphi,d})^{\perp_X} = \text{Ker } T_a^{\varphi,d}$ is a proper ideal of $\mathcal{O}_{X,a}$.*

Furthermore, if $D_{X,a}^{\psi,d}$ is D -invariant for some ψ , these conditions really hold for all local parametrisations φ at a and all the algebras $\mathcal{O}_{X,a}/(D_{X,a}^{\varphi,d})^{\perp_X}$ are canonically isomorphic for various φ .

Projectors T with the property that $\text{Ker } T$ are ideals play a major role in the interpolation theory since Birkhoff [Bi]. Equivalence of (1) and (2) is clearly stated in Marinari-Möller-Mora, [MMM] and de Boor-Ron [BR₂]. Let us call a point $a \in X$ *D -invariant of degree d* if it satisfies the condition (1) (or (2)) above for any φ and *D -invariant of degree ∞* if it is D -invariant of degree d for any $d \in \mathbb{N}$. The existence of the analytically open subset U with properties (1) follows from Theorem 4.2. We remark that the theorem above implies that the set of points which are not D -invariant of degree ∞ are contained in a countable union of thin analytic subsets of X , a set of first category in Baire’s sense with Lebesgue measure 0 in X .

The set of Bos-Calvi tangents gives an information on simplicity of embedding $X \subset \mathbb{C}^m$. Let us put

$$\theta_{\mathcal{O}_{n,b},\Phi}(d) := \max \left\{ \deg p : p \in \mathbb{C}[\Phi]_b^d \setminus \{0\} \right\}.$$

Let $\text{ord}_b(f)$ denote the vanishing order of f at b . Since $\theta_{\mathcal{O}_{n,b},\Phi}(d)$ is equal to

$$\sup \left\{ \text{ord}_b F(\Phi_1, \dots, \Phi_m) : F \in \mathbb{C}[\mathbf{x}]^d, F(\Phi_1, \dots, \Phi_m) \neq 0 \right\},$$

its estimate as a function of d is called a *zero estimates* of Φ . It is related to transcendence of the embedding of $X \subset \mathbb{C}^m$ at a . Historically, estimate of zero is fundamental in transcendental number theory. It is given for exponential polynomials, for some number theoretic functions or solutions of some good system of differential equations so far. Here we show that D -invariance

of degree ∞ implies an effective zero-estimate for a set of quite general holomorphic functions, but only at general points. Let

$$\chi(\bar{X}_a, d) := \dim_{\mathbb{C}} \mathbb{C}[\Phi]^d - \dim_{\mathbb{C}} \mathbb{C}[\Phi]^{d-1} = \dim_{\mathbb{C}} P^d(X_a) - \dim_{\mathbb{C}} P^{d-1}(X_a)$$

denote the Hilbert function of the smallest algebraic subset (the Zariski closure) \bar{X}_a of \mathbb{C}^m that contains a representative of the germ X_a . We have the following estimates.

Theorem 13.5. *Let $\Phi : \mathbb{C}_b \rightarrow X_a \subset \mathbb{C}_a^m$ be an embedding of a complex manifold. Then we have*

$$\binom{n+d}{n} + \theta_{O_{n,b}, \Phi}(d) - d \leq \dim_{\mathbb{C}} \mathbb{C}[\Phi]^d = \sum_{i=0}^d \chi(\bar{X}_a, i) \leq \binom{m+d}{m}$$

for any D -invariant point a of degree d . Hence the transcendency index

$$\alpha(X_a) := \limsup_{d \rightarrow \infty} \log_d \theta_{O_{n,b}, \Phi}(d),$$

defined in [Iz₂], is majorised by $\dim \bar{X}_a (\leq m)$ at D -invariant point of degree ∞ .

That is, the transcendency of an embedding $X \subset \mathbb{C}^m$ of complex manifold is not very high excepting points of a set expressed as a countable union of thin analytic subsets, even if X is quite general. This estimate has both merits and demerits in comparison with the zero-estimate obtained as a corollary of Gabrielov's multiplicity-estimate [Ga] of Noetherian functions on an integral manifold of a Noetherian vector field.

All results remain valid also for the real analytic category. We do not treat the multi-point interpolation problem as [BC₁] in this paper.

2. LEAST SPACE

Here we recall the least space of a vector space of holomorphic functions at a point. It is the graded space associated to the maximal-ideal-adic filtration. We use the term “least space” and the simple symbol \downarrow of de Boor and Ron [BR₁], [BR₂] in interpolation theory.

Let

$$O_{n,b} := \mathbb{C}\{t - b\} = \mathbb{C}\{t_1 - b_1, \dots, t_n - b_n\}$$

denote the local algebra of convergent power series centred at $b := (b_1, \dots, b_n) \in \mathbb{C}^n$ and

$$\mathfrak{m}_{n,b} := (t - b)O_{n,b} = (t_1 - b_1, \dots, t_n - b_n)O_{n,b}$$

its maximal ideal. This algebra $O_{n,b}$ has a filtration

$$O_{n,b} = \mathfrak{m}_{n,b}^0 \supset \mathfrak{m}_{n,b}^1 \supset \mathfrak{m}_{n,b}^2 \cdots$$

and it satisfies the following conditions:

$$\bigcap_{i \in \mathbb{N}_0} \mathfrak{m}_{n,b}^i = 0, \quad \dim_{\mathbb{C}} \frac{\mathfrak{m}_{n,b}^i}{\mathfrak{m}_{n,b}^{i+1}} = \frac{(n+i-1)!}{(n-1)! i!} < \infty.$$

We define the *least space* of $O_{n,b}$ by

$$O_{n,b} \downarrow := \bigoplus_{i \in \mathbb{N}_0} \frac{\mathfrak{m}_{n,b}^i}{\mathfrak{m}_{n,b}^{i+1}} \quad (\mathbb{N}_0 := \{0, 1, \dots\}).$$

An element contained in a single component $\mathfrak{m}_{n,b}^i/\mathfrak{m}_{n,b}^{i+1}$ is called *homogeneous*. Let us define the *order function*

$$\text{ord}_b : \mathcal{O}_{n,b} \longrightarrow \mathbb{N}_0, \quad f \longmapsto \text{ord}_b f;$$

by

$$\text{ord}_b f := \max\{i : f \in \mathfrak{m}_{n,b}^i\} \quad (\text{ord}_b 0 = +\infty)$$

and the *least operator*

$$\downarrow : \mathcal{O}_{n,b} \longrightarrow \mathcal{O}_{n,b}\downarrow, \quad f \longmapsto f_b\downarrow$$

by

$$f_b\downarrow := f \bmod \mathfrak{m}_{n,b}^{\text{ord}_b f + 1} \quad (0_b\downarrow := 0).$$

This least operator is not a linear map. We call $f_b\downarrow$ the *least part* of f . It is obvious that $\mathcal{O}_{n,b}\downarrow = \text{Span}(f_b\downarrow : f \in \mathcal{O}_{n,b})$, the linear span of the least parts of the elements of $\mathcal{O}_{n,b}$. The original definition of the least part $f_b\downarrow$ of f by de Boor and Ron is the non-zero homogeneous part of f of the smallest degree in the power series expansion of f with respect to some affine coordinate system. This homogeneous part is a coordinate expression of $f_b\downarrow$ defined above (see §8). Let us adopt the multi-exponent expression:

$$\mathbf{v} := (v_1, \dots, v_n), \quad (\mathbf{t} - \mathbf{b})^{\mathbf{v}} = (t_1 - b_1)^{v_1} \cdots (t_n - b_n)^{v_n}.$$

Since we consider elements of $\mathcal{O}_{n,b}\downarrow$ as belong to dual space of $\mathcal{O}_{n,b}$ in the later sections, we express them by polynomials in Greek variables as:

$$\tau_i := (t_i - b_i)_b\downarrow, \quad \boldsymbol{\tau}^{\mathbf{v}} := (\mathbf{t} - \mathbf{b})_b^{\mathbf{v}}\downarrow \in \mathfrak{m}_{n,b}^{|\mathbf{v}|}/\mathfrak{m}_{n,b}^{|\mathbf{v}|+1}.$$

Hence the least space $\mathcal{O}_{n,b}\downarrow$ is denoted by the polynomial algebra $\mathbb{C}[\boldsymbol{\tau}]$ with $\boldsymbol{\tau} := (\tau_1, \dots, \tau_n)$, $\tau_i := (t_i - b_i)_b\downarrow$. The product in $\mathbb{C}[\boldsymbol{\tau}]$ is a natural operation as a consequence of the property

$$\mathfrak{m}_{n,b}^i \mathfrak{m}_{n,b}^j = \mathfrak{m}_{n,b}^{i+j}.$$

Let Z_b be a vector subspace of $\mathcal{O}_{n,b}$. We put

$$Z_b\downarrow := \text{Span}(f_b\downarrow : f \in Z_b) \subset \mathbb{C}[\boldsymbol{\tau}],$$

the linear span of $\{f_b\downarrow : f \in Z_b\}$ and call this the *least space* of Z_b . We know the following (which will be strengthened to Theorem 5.4).

Theorem 2.1 (de Boor-Ron [BR₂], Proposition 2.10; cf. [Iz₃], Theorem 7.1). *Let Z_b be a finite-dimensional vector subspace of $\mathcal{O}_{n,b}$. Then we have*

$$\dim_{\mathbb{C}} Z_b\downarrow = \dim_{\mathbb{C}} Z_b.$$

Proposition 2.2. *If $0 < \dim_{\mathbb{C}} Z < \infty$ and $Z_b\downarrow$ is D -invariant, then there exists $f \in Z$ such that $f(\mathbf{b}) \neq 0$.*

Proof. If $f(\mathbf{b}) = 0$ for all $f \in Z$, we have $Z_b \subset \mathfrak{m}_{n,b}$. Then $Z_b\downarrow$ is not D -invariant, contradiction. \square

Therefore no point of the simultaneous vanishing locus of the elements of Z has D -invariant $Z_b\downarrow$. The converse does not hold as we will see in Example 7.7.

3. JET SPACES AND MULTIVARIATE WRONSKIANS

Let Z be a vector space of holomorphic functions on an open subset $U \subset \mathbb{C}^n$. The k -th jets of the elements of Z at a point of U form a vector space. If we gather such vector spaces only at good points of U , we get a vector bundle. The theorem of Walker on Wronskian gives the least set of candidates of fibre coordinates for general Z .

Let \mathcal{O}_n denote the sheaf of germs of holomorphic functions on \mathbb{C}^n . We call the sheaf of germs of holomorphic sections of a holomorphic vector bundle the *associated sheaf* and express it by the script style of the same letter expressing the bundle. The correspondence of vector bundles on U to the associated sheaves defines a bijective mapping of the set of isomorphism classes of vector bundles of rank r over U onto the set of isomorphism classes of locally free analytic sheaves (sheaves of \mathcal{O}_U -modules, $\mathcal{O}_U = \mathcal{O}_n|_U$) of rank r over U (see e.g. [PR], Proposition 3.3). In the below, note that the parenthesised \mathbf{b} means the values or the sets of the values at \mathbf{b} (bundle fibre, e.g. $R_k(\mathbf{b})$), while \mathbf{b} in subscript style means the germs or the sets of the germs at $\mathbf{b} \in U$ (stalk, sheaf fibre, e.g. $\mathcal{L}_{\mathbf{b}}^k, T_{\mathbf{b}}^*$) or the indication of the attention point of the least operator (as $Z_{\mathbf{b}} \downarrow$).

Proposition 3.1. *If $Z \subset \mathbb{C}[\tau]$ is D -invariant, it is translation invariant i.e. $p(\tau) \in Z$ implies $p(\tau + \mathbf{b}) \in Z$ for any constant vector \mathbf{b} .*

Proof. This is obvious from the ordinary Taylor formula:

$$p(\tau + \mathbf{b}) = \sum_{|\mathbf{v}| \leq d} \frac{1}{\mathbf{v}!} \frac{\partial^{|\mathbf{v}|} p(\tau)}{\partial \tau^{\mathbf{v}}} \mathbf{b}^{\mathbf{v}} \quad (d := \deg p).$$

□

Let

$$\pi_k : J^k(\mathcal{O}_U) \longrightarrow U \quad \left(J^k(\mathcal{O}_U) \cong U \times \mathbb{C}^{N(n,k)}, N(n,k) := \binom{n+k}{k} \right)$$

denote the k -th jet space of holomorphic functions on U , the holomorphic vector bundle of k -th jets of holomorphic functions defined on open subsets of U . Its coordinates are denoted by

$$(\mathbf{t}, (u_{\mathbf{v}} : |\mathbf{v}| \leq k)) \quad (\mathbf{t} := (t_1, \dots, t_n) \in U).$$

The coordinates $u_{\mathbf{v}}$ ($|\mathbf{v}| \leq k$) are called the *fibre coordinates* corresponding to the \mathbf{v} -th derivative. Let $\mathcal{O}_n(V)$ denote the algebra of sections of \mathcal{O}_n over $V \subset U$. The k -th jet extension $j^k f$ of $f \in \mathcal{O}_n(V)$ is defined by

$$j^k f : V \longrightarrow J^k(\mathcal{O}_U), \quad \mathbf{t} \longmapsto (\mathbf{t}, u_{\mathbf{v}}(j^k f) : |\mathbf{v}| \leq k) \\ \left(u_{\mathbf{v}}(j^k f) = \frac{1}{\mathbf{v}!} \frac{\partial^{|\mathbf{v}|} f(\mathbf{t})}{\partial \mathbf{t}^{\mathbf{v}}} \right).$$

This is a section of the jet space $J^k(\mathcal{O}_U)$ over V . The normalising factor $1/\mathbf{v}!$ of the \mathbf{v} -th fibre coordinate works in the calculation of prolongation below.

If Z is a finite-dimensional vector subspace of $\mathcal{O}_n(U)$, the evaluation of the jet extension at $\mathbf{b} \in U$ defines the mapping

$$j^k|_Z(\mathbf{b}) : Z \longrightarrow J^k(\mathcal{O}_U)(\mathbf{b}), \quad f \longmapsto j^k f(\mathbf{b}).$$

Let

$$(\mathbf{b}, R_k(\mathbf{b})) := \{j^k f(\mathbf{b}) : f \in Z\}$$

denote its image. Then we have the natural commutative Diagram 1 of linear mappings of vector spaces. Here, the horizontal mappings are projections defined by forgetting the coordinates

$$\begin{array}{ccccc} & & \text{DIAGRAM 1 } Z & \text{Jet spaces of } Z \subset \mathbb{C}[\tau] & \\ & \swarrow j^{k-1}|_Z(\mathbf{b}) & \downarrow j^k|_Z(\mathbf{b}) & \searrow j^{k+1}|_Z(\mathbf{b}) & \\ \cdots \longleftarrow & R_{k-1}(\mathbf{b}) & \xleftarrow{\Sigma^k(\mathbf{b})} R_k(\mathbf{b}) & \xleftarrow{\Sigma^{k+1}(\mathbf{b})} R_{k+1}(\mathbf{b}) & \longleftarrow \cdots \end{array}$$

corresponding to the highest order derivatives. The total image $\bigcup_{\mathbf{b} \in U} (\mathbf{b}, R_k(\mathbf{b}))$ is not an analytic subset of $J^k(\mathcal{O}_U)$ nor even a closed subset in general. We say the complement in U of a closed analytic subset of U *analytically open* in U . Let us put

$$r_k := \max\{\dim_{\mathbb{C}} R_k(\mathbf{t}) : \mathbf{t} \in U\}, \quad U_k := \{\mathbf{t} \in U : \dim_{\mathbb{C}} R_k(\mathbf{t}) = r_k\}.$$

Suppose that U is connected. Since the points of U_k are characterised by the full rank condition of certain matrices with holomorphic elements, U_k is a non-empty analytically open subset. Putting

$$R_k := \{(\mathbf{b}, R_k(\mathbf{b})) : \mathbf{b} \in U_k\},$$

we have a holomorphic vector sub-bundle

$$\pi_k|_{R_k} : R_k \longrightarrow U_k$$

of $J^k(\mathcal{O}_U)|_{U_k}$.

Definition 3.2. We call a point of U_k a *bundle point of the k -th jet space of Z* .

Now we recall Walker's theorem on multivariate Wronskians. It allows us to write out a finite system of PDEs explicitly whose solution space is a given finite-dimensional vector subspace $Z \subset \mathcal{O}_n(U)$ and it enables us to state the subsequent arguments efficiently.

Definition 3.3. Walker called $Y := \{\nu_1, \dots, \nu_m\} \subset \mathbb{N}_0^n$ *Young-like* if it satisfies the following condition:

$$(\nu \in \mathbb{N}_0^n, \exists \nu_i \in Y : \nu \leq \nu_i) \implies \nu \in Y.$$

Here \leq implies the order obtained as the product of the usual order \leq of \mathbb{N}_0 . The property of Young-likeness is equivalent to the condition that $\text{Span}(\tau^{\nu_1}, \dots, \tau^{\nu_m})$ is D -invariant. Let us put

$$\mathcal{Y}_m := \{\{\nu_1, \dots, \nu_m\} \subset \mathbb{N}_0^n : \text{Young-like}\}.$$

Theorem 3.4 (Walker [Wa], Theorem 3.1). *Let U be a connected open subset of \mathbb{C}^n . Let $\{f_1, \dots, f_m\}$ be elements of a vector space of meromorphic functions on U . Then the functions*

f_1, \dots, f_m are linearly independent if and only if there exists at least one $\{\mathbf{v}_1, \dots, \mathbf{v}_m\} \in \mathcal{Y}_m$ such that

$$W(f_1, \dots, f_m; \mathbf{v}_1, \dots, \mathbf{v}_m) := \begin{vmatrix} f_1^{(\mathbf{v}_1)} & f_1^{(\mathbf{v}_2)} & \dots & f_1^{(\mathbf{v}_m)} \\ f_2^{(\mathbf{v}_1)} & f_2^{(\mathbf{v}_2)} & \dots & f_2^{(\mathbf{v}_m)} \\ \vdots & \vdots & \ddots & \vdots \\ f_m^{(\mathbf{v}_1)} & f_m^{(\mathbf{v}_2)} & \dots & f_m^{(\mathbf{v}_m)} \end{vmatrix}$$

does not vanish identically. The set \mathcal{Y}_m is minimal among the sets with this property.

We need the following immediate consequence of this theorem.

Corollary 3.5 (to Walker's theorem). *Let $\{f_1, \dots, f_m\}$ be a basis of a vector subspace $Z \subset O_n(U)$. Then Z is the space of holomorphic solutions of the system of PDEs:*

$$W(f_1, \dots, f_m, y; \mathbf{v}_1, \dots, \mathbf{v}_{m+1}) = 0 \quad (\{\mathbf{v}_1, \dots, \mathbf{v}_{m+1}\} \in \mathcal{Y}_{m+1}),$$

where $y = y(\mathbf{t}) \in O_n(U)$ denotes the unknown function.

Definition 3.6. Let $\{f_1, \dots, f_m\}$ be a basis of a vector subspace $Z \subset O_n(U)$. Let us call a point where the evaluation of $W(f_1, \dots, f_m, y; \mathbf{v}_1, \dots, \mathbf{v}_m)$ for some $\{\mathbf{v}_1, \dots, \mathbf{v}_m\} \in \mathcal{Y}_m$ does not vanish a *bundle point* of all the jet spaces of Z . The set of the bundle points of all the jet spaces of Z is denoted by U_Z^{bdl} .

Corollary 3.7 (to Walker's theorem). *Let $\{f_1, \dots, f_m\}$ be a basis of a vector subspace $Z \subset O_n(U)$. The sets U_k of bundle points of the k -th jet space of Z and the set U_Z^{bdl} of the bundle points of all the jet spaces of Z are non-empty analytically open subset of U and independent of the change of local coordinates of U for fixed Z . The higher derivatives of $\{f_1, \dots, f_m\}$ of orders appearing in a non-vanishing Wronskian spans all other ones on all over U . It means that such derivatives form a fibre coordinates of R_{m-1} around \mathbf{b} . This implies that*

$$U_Z^{\text{bdl}} = U_0 \cap U_1 \cap \dots = U_0 \cap \dots \cap U_{m-1}.$$

Proof. Since $\dim_{\mathbb{C}} \text{Span}((f_1^{\mathbf{v}}, \dots, f_m^{\mathbf{v}}) : \mathbf{v} \in \mathbb{N}_0^m) = m$, The higher derivatives of $\{f_1, \dots, f_m\}$ of orders appearing in a non-vanishing Wronskian spans all other ones on U . Openness of U_k is stated above. If $\{\mathbf{v}_1, \dots, \mathbf{v}_m\}$ is Young-like we have $|\mathbf{v}_i| \leq m-1$ ($1 \leq i \leq m$), which implies the last equality. Hence openness of U_Z^{bdl} holds. Since jet spaces are contravariant geometric object, independence from coordinate change is obvious. \square

4. D -INVARIANCE OF THE LEAST SPACES

Let Z be a vector space of holomorphic functions on an open subset $U \subset \mathbb{C}^n$. The vector space of the germs of the elements of Z at $\mathbf{b} \in U$ is denoted by $Z_{\mathbf{b}}$. The least space $Z_{\mathbf{b}} \downarrow$ of $Z_{\mathbf{b}}$ is identified as a vector subspace of $\mathbb{C}[\tau]$ (as stated in Introduction). Our main purpose here is to prove that $Z_{\mathbf{b}} \downarrow$ is D -invariant, namely it is invariant with respect to all the partial differentiations $\partial/\partial \tau_i$, at a general point of U . The point of the proof is the prolongation of PDEs annihilating the jets of elements of Z .

Let \mathcal{L}^k denote the sheaf of germs of holomorphic functions on $J^k(\mathcal{O}_{U_Z^{\text{bdl}}})$ vanishing on R_k which are linear in fibre coordinates u_ν . The local sections of \mathcal{L}^k are functions

$$\varphi := \sum_{|\nu| \leq k} \alpha_\nu(\mathbf{t}) \cdot u_\nu$$

which are homogeneous linear in fibre coordinates u_ν of $J^k(\mathcal{O}_{U_Z^{\text{bdl}}})$ with coefficients $\alpha_\nu(\mathbf{t}) \in \mathcal{O}_n(V)$ ($V \subset U_Z^{\text{bdl}}$). If u_ν are replaced by the corresponding differential operators $\partial^{|\nu|}/\partial \mathbf{t}^\nu$, then elements of $\mathcal{L}^k(V)$ become homogeneous linear partial differential operators which annihilates the functions of Z on V .

Definition 4.1. Take a local section

$$\varphi := \sum_{|\nu| \leq k-1} \alpha_\nu(\mathbf{t}) \cdot u_\nu \in \mathcal{L}^{k-1}(V)$$

over $V \subset U$. Differentiating the relation

$$\sum_{|\nu| \leq k-1} \frac{1}{\nu!} \alpha_\nu(\mathbf{t}) \cdot \frac{\partial^{|\nu|} f}{\partial \mathbf{t}^\nu} = 0 \quad (f \in Z)$$

by t_i , we have

$$\sum_{|\nu| \leq k-1} \frac{1}{\nu!} \left(\alpha_\nu^{(e_i)}(\mathbf{t}) \cdot \frac{\partial^{|\nu|}}{\partial \mathbf{t}^\nu} + \alpha_\nu(\mathbf{t}) \cdot \frac{\partial^{|\nu+e_i|}}{\partial \mathbf{t}^{\nu+e_i}} \right) f = 0 \quad (f \in Z),$$

where $\mathbf{e}_i := (\delta_{1i}, \delta_{2i}, \dots, \delta_{ni})$ denotes the i -th unit vector. Hence $\alpha_\nu^{(e_i)}$ expresses the derivative by t_i . Thus we have proved that

$$\sum_{|\nu| \leq k-1} \left(\alpha_\nu^{(e_i)}(\mathbf{t}) \cdot u_\nu + (\nu_i + 1) \alpha_\nu(\mathbf{t}) \cdot u_{\nu+e_i} \right) \in \mathcal{L}^k(V) \quad (i = 1, \dots, n).$$

These are called the *first prolongations* of φ .

Theorem 4.2. Let $\mathcal{O}_n(U)$ be the algebra of holomorphic functions on a connected open subset $U \subset \mathbb{C}^n$, Z its finite-dimensional vector subspace and $U_Z^{\text{bdl}} \subset U$ the set of bundle points of all the jet spaces of Z (Definition 3.6). Then the least space $Z_{\mathbf{b}} \downarrow \subset \mathbb{C}[\tau]$ of Z at \mathbf{b} is D -invariant for every point $\mathbf{b} \in U_Z^{\text{bdl}}$.

Proof. The canonical projections

$$\Pi^k : J_k(\mathcal{O}_U) \longrightarrow J_{k-1}(\mathcal{O}_U), \quad \Sigma^k : R_k \longrightarrow R_{k-1}$$

are constant rank homomorphisms on U_Z^{bdl} and they induce the inclusion $i^k : \mathcal{Ker} \Sigma^k \longrightarrow \mathcal{Ker} \Pi^k$ of locally free analytic sheaves by the following diagram:

$$\begin{array}{ccccccc} \mathcal{R}_{k-1} & \xleftarrow{\Sigma^k} & \mathcal{R}_k & \xleftarrow{\quad} & \mathcal{Ker} \Sigma^k & \xleftarrow{\quad} & 0 \\ \downarrow & & \downarrow & & \downarrow i^k & & \\ \mathcal{J}_{k-1}(\mathcal{O}_U) & \xleftarrow{\Pi^k} & \mathcal{J}_k(\mathcal{O}_U) & \xleftarrow{\quad} & \mathcal{Ker} \Pi^k & \xleftarrow{\quad} & 0. \end{array}$$

A local section of $\mathcal{Ker} \Pi^k$ is expressed as

$$f(\mathbf{t}, \boldsymbol{\tau}) = \sum_{|\mathbf{v}|=k} \beta_{\mathbf{v}}(\mathbf{t}) \boldsymbol{\tau}^{\mathbf{v}} \quad \left(\beta_{\mathbf{v}} := \frac{1}{\mathbf{v}!} \frac{\partial^{|\mathbf{v}|} f(\mathbf{t}, \mathbf{0})}{\partial \boldsymbol{\tau}^{\mathbf{v}}} \right),$$

where the monomial $\boldsymbol{\tau}^{\mathbf{v}}$ stands for the base of the fibre corresponding to coordinate $u_{\mathbf{v}}$. We may write $\boldsymbol{\tau}^{\mathbf{v}}$ as $(d\mathbf{t})^{\odot \mathbf{v}} = dt_1^{v_1} \cdots dt_n^{v_n}$ (see Theorem 8.1, (1)).

The least space $Z_{\mathbf{b}} \downarrow$ is got by evaluating $\mathcal{Ker} \Sigma^k$ at \mathbf{b} . Hence D -invariance of $Z_{\mathbf{b}} \downarrow$ at degree k reduces to the implication

$$f \in \mathcal{Ker} \Sigma^k \implies \frac{\partial f}{\partial \tau_i} \in \mathcal{Ker} \Sigma^{k-1}.$$

Take $f(\mathbf{t}, \boldsymbol{\tau}) \in \mathcal{Ker} \Sigma^k$ and any defining equation

$$\varphi := \sum_{|\mathbf{v}| \leq k-1} \alpha_{\mathbf{v}}(\mathbf{t}) \cdot u_{\mathbf{v}} \in \mathcal{L}^{k-1}(V)$$

of \mathcal{R}_{k-1} over a neighbourhood V of \mathbf{b} , we have

$$\begin{aligned} \varphi \left(j^k \left(\frac{\partial f}{\partial \tau_i} \right) \right) &= \sum_{|\mathbf{v}| \leq k-1} \alpha_{\mathbf{v}}(\mathbf{t}) \cdot u_{\mathbf{v}} \left(j^k \left(\frac{\partial f}{\partial \tau_i} \right) \right) \\ &= \sum_{|\mathbf{v}| \leq k-1} (v_i + 1) \alpha_{\mathbf{v}}(\mathbf{t}) \cdot u_{\mathbf{v} + \mathbf{e}_i} (j^k f) \\ &= \sum_{|\mathbf{v}| \leq k-1} \left(\alpha_{\mathbf{v}}^{(\mathbf{e}_i)}(\mathbf{t}) \cdot u_{\mathbf{v}} + (v_i + 1) \alpha_{\mathbf{v}}(\mathbf{t}) \cdot u_{\mathbf{v} + \mathbf{e}_i} \right) (j^k f) = 0. \end{aligned}$$

Here, since $|\mathbf{v}| \leq k-1$ implies $u_{\mathbf{v}}(j^k f) = 0$, the third equality follows. The last equality, because of the first prolongations stated above. This proves that $\partial f / \partial \tau_i \in \mathcal{R}_{k-1, \mathbf{b}}$. The inclusion $\partial f / \partial \tau_i \in \mathcal{Ker} \Sigma_{\mathbf{b}}^{k-1}$ follows from $f(\mathbf{t}, \boldsymbol{\tau}) \in \mathcal{Ker} \Sigma^k$ (homogeneity of f). Evaluating at $\mathbf{b} \in U_k \cap U_{k-1}$, we have shown that $Z_{\mathbf{b}} \downarrow$ is D -invariant at degree k . Then total $Z_{\mathbf{b}} \downarrow$ is D -invariant on the open subset

$$U_Z^{\text{bdl}} := U_0 \cap U_1 \cap \cdots = U_0 \cap \cdots \cap U_{m-1}$$

described in Proposition 3.7. \square

By Corollary 3.5, there exists $k \in \mathbb{N}_0$ such that a system of linear PDEs of order $k+1$ is sufficient to select the sections of Z , namely Z is defined by \mathcal{L}^k . We may call the sheaf \mathcal{L}^k for such k is the *defining system of PDEs of order k* for Z . Since Corollary 3.5 assures the existence of $k \leq m-1$, \mathcal{L}^{m-1} is a defining system of Z .

5. SESQUILINEAR FORMS AND WEAK TOPOLOGIES

Here we recall the standard sesquilinear form on the product $\mathbb{C}[\boldsymbol{\tau}] \times \mathbb{C}\{\mathbf{t}\}$ of the space of polynomials and the convergent power series algebra. The restriction of this sesquilinear form to products of a finite dimensional subspace $Z \subset \mathbb{C}\{\mathbf{t}\}$ and its least space $Z_{\mathbf{b}} \downarrow \subset \mathbb{C}[\boldsymbol{\tau}]$ is proved to be non-degenerate by de Boor-Ron [BR₁], [BR₂]. Although most of the assertions on sesquilinear forms below are seemingly obvious, we decide the notation and certify their validity through the definite statements on bilinear forms in Bourbaki [Bo₁].

Let us define a complex bilinear form

$$B_{n,b} : \mathbb{C}[\tau] \times O_{n,b} \longrightarrow \mathbb{C}, \quad (p, f) \longmapsto B_{n,b} \langle p \parallel f \rangle,$$

by

$$B_{n,b} \left\langle \sum_{\text{finite}} a_\nu \tau^\nu \parallel \sum b_\mu (t - b)^\mu \right\rangle := \sum_{\text{finite}} \nu! a_\nu b_\nu,$$

where $\nu! = \nu_1! \cdots \nu_m!$. In particular,

$$B_{n,b} \langle \tau^\nu \parallel (t - b)^\mu \rangle = \frac{\partial^{|\nu|} (t - b)^\mu}{\partial t^\nu} (b) = \frac{\partial^{\nu_1 + \cdots + \nu_n} (t - b)^\mu}{\partial t_1^{\nu_1} \cdots \partial t_n^{\nu_n}} (b) = \begin{cases} \nu! & (\nu = \mu) \\ 0 & (\nu \neq \mu) \end{cases}.$$

Thus the monomial τ^ν can be identified with the signed higher order derivative $(-1)^{|\nu|} \delta_b^{(\nu)}$ of the Dirac delta function supported at $b \in \mathbb{C}^m$. The sign $(-1)^{|\nu|}$ here originates from partial integral in the Schwartz distribution theory [Sc], II,3;35.

Now let u denote the complex conjugations:

$$\begin{aligned} u : \mathbb{C}[\tau] &\longrightarrow \mathbb{C}[\tau], & \sum_{\text{finite}} a_\nu \tau^\nu &\longmapsto \sum_{\text{finite}} \bar{a}_\nu \tau^\nu, \\ u : O_{n,b} &\longrightarrow O_{n,b}, & \sum b_\mu (t - b)^\mu &\longmapsto \sum \bar{b}_\mu (t - b)^\mu. \end{aligned}$$

The sesquilinear form

$$S_{n,b} : \mathbb{C}[\tau] \times O_{n,b} \longrightarrow \mathbb{C}, \quad (p, f) \longmapsto S_{n,b} \langle p \mid f \rangle$$

is defined by

$$S_{n,b} \langle p \mid f \rangle := B_{n,b} \langle p \parallel u(f) \rangle.$$

This can be expressed also as

$$S_{n,b} := B_{n,b} \circ (\text{id}_{\mathbb{C}[\tau]}, u).$$

Explicitly, we have

$$S_{n,b} \left\langle \sum_{\text{finite}} a_\nu \tau^\nu \mid \sum b_\mu (t - b)^\mu \right\rangle = \sum \nu! a_\nu \bar{b}_\nu.$$

The *weak topology* of $\mathbb{C}[\tau]$ with respect to $B_{n,b}$ is the coarsest topology such that the linear functionals

$$b_{\parallel f} : \mathbb{C}[\tau] \longrightarrow \mathbb{C}, \quad p \longmapsto B_{n,b} \langle p \parallel f \rangle$$

are continuous for all $f \in O_{n,b}$. Similarly the weak topology of $O_{n,b}$ with respect to $B_{n,b}$ is the coarsest topology such that the linear functionals

$$t_{p\parallel} : O_{n,b} \longrightarrow \mathbb{C}, \quad f \longmapsto B_{n,b} \langle p \parallel f \rangle$$

are continuous for all $p \in \mathbb{C}[\tau]$. By these topologies, $\mathbb{C}[\tau]$ and $O_{n,b}$ become topological vector spaces. If L is a vector subspace of $\mathbb{C}[\tau]$, its annihilator space (orthogonal space) with respect to $B_{n,b}$ is denoted by L^\top :

$$L^\top := \{f : B_{n,b} \langle p \parallel f \rangle = 0 \text{ for all } p \in L\}.$$

Similarly, if K is a vector subspace of $O_{n,b}$, its annihilator space with respect to $B_{n,b}$ is denoted by K^\top :

$$K^\top := \{p : B_{n,b} \langle p \parallel f \rangle = 0 \text{ for all } f \in K\}.$$

These are vector subspaces. We adopt the abbreviation $L^{\top\top} = (L^\top)^\top$ and $K^{\top\top} = (K^\top)^\top$, etc. Recall that $L^{\top\top}$ and $K^{\top\top}$ are the weak closures of L and K respectively (Bourbaki [Bo₂], 4, §1, n°3, 4).

The *weak topologies* of $\mathbb{C}[\tau]$ and $O_{n,b}$ with respect to $S_{n,b}$ are defined in a similar way to those with respect $B_{n,b}$ and they become topological vector spaces. The weak topology of $O_{n,b}$ is nothing but that of the coefficient-wise convergence. If $L \subset \mathbb{C}[\tau]$ and $K \subset O_{n,b}$ are vector subspaces, their annihilators spaces with respect to $S_{n,b}$ are denoted by L^\perp , K^\perp respectively:

$$\begin{aligned} L^\perp &:= \{f : S_{n,b} \langle p | f \rangle = 0 \text{ for all } p \in L\}, \\ K^\perp &:= \{p : S_{n,b} \langle p | f \rangle = 0 \text{ for all } f \in K\}. \end{aligned}$$

These are also vector subspaces. We adopt the abbreviation $L^{\perp\perp} = (L^\perp)^\perp$ and $K^{\perp\perp} = (K^\perp)^\perp$, etc. It is obvious that

$$\begin{aligned} S_{n,b} \langle p | f \rangle &= B_{n,b} \langle p | u(f) \rangle = \overline{B_{n,b} \langle u(p) | f \rangle}, \\ B_{n,b} \langle p | f \rangle &= S_{n,b} \langle p | u(f) \rangle = \overline{S_{n,b} \langle u(p) | f \rangle}. \end{aligned}$$

Defining $s_{|f}$, $s_{|p|}$ in a similar way as $b_{|f}$, $b_{|p|}$, we have

$$s_{|u(f)} = b_{|f}, \quad s_{|f} = b_{|u(f)}, \quad s_{|p|} = t_{|p|} \circ u, \quad t_{|p|} = s_{|p|} \circ u.$$

Since u is an involution (i.e. $u \circ u$ is the identity) and commute with the operations $^\perp$ and $^\top$, we have the following.

Proposition 5.1. *The weak topologies on $\mathbb{C}[\tau]$ (resp. $O_{n,b}$) with respect to $B_{n,b}$ and $S_{n,b}$ coincide. If L (resp. K) is a vector subspace of $\mathbb{C}[\tau]$ (resp. $O_{n,b}$), we have the natural isomorphisms*

$$\begin{aligned} L^\perp &= u(L^\top) = (u(L))^\top, & K^\top &= u(K^\perp) = (u(K))^\perp, \\ K^\perp &= u(K^\top) = (u(K))^\top, & L^\top &= u(L^\perp) = (u(L))^\perp \end{aligned}$$

and hence

$$L^{\perp\perp} = L^{\top\top}, \quad K^{\top\top} = K^{\perp\perp}.$$

By the first assertion, we have no need to refer to forms $B_{n,b}$ and $S_{n,b}$ for weak topologies. Now we can deduce a few properties on the space of annihilators with respect to the sesquilinear forms from those with respect to bilinear forms (Bourbaki [Bo₁], 4, §1, n°4).

Corollary 5.2. *If L (resp. K) is a vector subspace of $\mathbb{C}[\tau]$ (resp. $O_{n,b}$), we have the following.*

- (1) K^\perp , L^\perp are all weakly closed.
- (2) $L^{\perp\perp}$ is the weak closure of L in $\mathbb{C}[\tau]$ and $K^{\perp\perp}$ is the weak closure of K in $O_{n,b}$.
- (3) $L^{\perp\perp\perp} = L^\perp$, $K^{\perp\perp\perp} = K^\perp$.

We can easily prove the following.

Proposition 5.3. *The form*

$$S_{n,b} : \mathbb{C}[\tau] \times O_{n,b} \longrightarrow \mathbb{C}$$

is a non-degenerate sesquilinear form, i.e. $\mathbb{C}[\tau]^\perp = \{0\}$ and $O_{n,b}^\perp = \{0\}$.

Let us follow the notation in §2:

$$\tau^\nu := (t - b)_{b\downarrow}^\nu \in \mathfrak{m}_{n,b}^{|\nu|} / \mathfrak{m}_{n,b}^{|\nu|+1}.$$

The following is fundamental to construct a dual space of a finite dimensional subspace of $O_{n,b}$.

Theorem 5.4 (de Boor-Ron [BR₂], Theorem 5.8). *Let Z be a vector subspace of $O_{n,b}$. Then the sesquilinear form*

$$S_Z : Z_b \downarrow \times Z \longrightarrow \mathbb{C}$$

obtained as the restriction of $S_{n,b} : \mathbb{C}[\tau] \times O_{n,b} \longrightarrow \mathbb{C}$ is non-degenerate, i.e. $\mathbb{C}[\tau]^{\perp_Z} = \{0\}$, $O_{n,b}^{\perp_Z} = \{0\}$, where \perp_Z denotes the space of annihilators with respect to S_Z .

Proof. Suppose that $f \neq 0$ belongs to Z and annihilates $Z_b \downarrow$. We can express f as

$$f = \sum_{|\nu| \geq d} a_\nu \tau^\nu \quad (d := \text{ord}_b f < \infty).$$

Then we have

$$0 = S_Z(f_b \downarrow, f) = \sum_{|\nu|=d} \nu! |a_\nu|^2 \neq 0,$$

a contradiction. This proves that the space of annihilators $(Z_b \downarrow)^{\perp_Z} = \{0\}$.

Suppose that $p \neq 0$ belongs to $Z_b \downarrow$ and annihilates Z . Let $p_b \uparrow \neq 0$ denote the highest degree homogeneous part of p at b . Since $Z_b \downarrow$ is generated by homogeneous elements, there exists $g \in Z$ such that $p_b \uparrow = g_b \downarrow$. Then

$$0 < S_n \langle p_b \uparrow | p_b \uparrow \rangle = S_n \langle p_b \uparrow | g_b \downarrow \rangle = S_Z \langle p | g \rangle = 0,$$

a contradiction. This proves that $Z^\perp = \{0\}$. \square

Remark 5.5. A change of affine coordinates of \mathbb{C}^n amounts to change of the matrix expressing the form from real diagonal to Hermitian but the weak topology remains unchanged.

Lemma 5.6. *Multiplication by $p(\tau)$ in $\mathbb{C}[\tau]$ is the adjoint of applying the differential operator $u(p)(\partial_t)$ ($\partial_t := (\partial/\partial t_1, \dots, \partial/\partial t_n)$) on $O_{n,b}$, with respect to $S_{n,b}$. In particular, multiplication by τ_i is the adjoint of Differentiation by t_i . Similarly, multiplication by $f(t)$ in $O_{n,b}$ is the adjoint of applying the infinite differential operator $u(f)(\partial_\tau)$ ($\partial_\tau := (\partial/\partial \tau_1, \dots, \partial/\partial \tau_n)$) on $\mathbb{C}[\tau]$, with respect to $S_{n,b}$. (Such an operator is valid only for polynomials.) In particular, multiplication by t_i is the adjoint of Differentiation by τ_i .*

Proof. The second assertion is obvious from the direct calculations:

$$\begin{aligned} S_{n,b} \left\langle \sum_\lambda c_\lambda \partial_\tau^\lambda \sum_\nu a_\nu \tau^\nu \middle| \sum_\mu b_\mu t^\mu \right\rangle &= \sum_{\nu, \lambda} (\lambda + \nu)! c_\lambda a_{\lambda+\nu} \bar{b}_\nu \\ &= \sum_\nu \sum_{\mu \geq \nu} \mu! a_\mu \bar{b}_\nu c_{\mu-\nu} = S_{n,b} \left\langle \sum_\nu a_\nu \tau^\nu \middle| \sum_\lambda \bar{c}_\lambda t^\lambda \sum_\mu b_\mu t^\mu \right\rangle, \end{aligned}$$

where \geq denotes the product order of the order of the set of integers. The proof of the first inequality is quite similar. \square

6. CONTINUITY OF HOMOMORPHISMS OF ANALYTIC ALGEBRAS

Her we recall a few topological properties of the homomorphisms of analytic algebras. The main reference is Grauert-Remmert [GR].

An algebra isomorphic to a factor algebra of a convergent power series algebra by a proper ideal is called (*local*) *analytic algebra*. Take an analytic algebra $A := O_{n,b}/I$. This is a local

\mathbb{C} -algebra in the sense it has a unique maximal ideal \mathfrak{m}_A , which consists of the residue classes of elements of $\mathfrak{m}_{n,b}$, and A is a vector space over the subalgebra $\mathbb{C} \subset A$ such that the canonical homomorphism

$$\mathbb{C} \longrightarrow A/\mathfrak{m}_A, \quad \lambda \longmapsto \lambda \cdot 1 \bmod \mathfrak{m}_A$$

of the fields is an isomorphism.

The sesquilinear form $S_{n,b} : \mathbb{C}[\tau] \times \mathcal{O}_{n,b} \longrightarrow \mathbb{C}$ induces a non-degenerate one $S_A : I^{\perp_n} \times A \longrightarrow \mathbb{C}$ by Bourbaki [Bo₁], 4, §1, n°5. We can define a *weak topology* on A using this S_A . Following Grauert-Remmert [GR], let us see in the following lemma that this weak topology on A is independent of the expression $A = \mathcal{O}_{n,b}/I$. Let $\pi_i : A \longrightarrow A/\mathfrak{m}_A^i$ ($i \in \mathbb{N}$) be the factor epimorphism. The analytic algebra A/\mathfrak{m}_A^i is a finite-dimensional \mathbb{C} -vector space and hence it has a unique structure of a topological vector space. We give A the coarsest topology such that all the π_i ($i \in \mathbb{N}$) are continuous and call it the *projective topology*. Of course this is independent of the expression $\mathcal{O}_{n,b}/I$.

Lemma 6.1. *The projective (resp. weak) topology of $A := \mathcal{O}_{n,b}/I$ coincides with the topology of quotient of $\mathcal{O}_{n,b}$ with the projective (resp. weak) topology. Consequently, the weak topology and the projective topology on A coincide and the factor epimorphism $\mathcal{O}_{n,b} \longrightarrow A$ is always a weakly continuous and open mapping.*

Proof. It is easy to see that the projective topology and the weak topology coincide on a regular analytic algebra $\mathcal{O}_{n,b}$. The projective topology on A is proved to coincide with the topology of quotient of $\mathcal{O}_{n,b}$ by Grauert-Remmert [GR], Satz II.1.3. Since the ideal $I \subset \mathcal{O}_{n,b}$ is closed with respect to the projective topology by [GR], Satz II.1.2, it is closed also with respect to the weak topology and we have $I = I^{\perp_n \perp_n}$. Then, applying [Bo₁], 4, §1, n°5 to the non-degenerate sesquilinear form S_A on $I^{\perp_n} \times A \cong I^{\perp_n} \times (\mathcal{O}_{n,b}/I^{\perp_n \perp_n})$, we see that the weak topology on A coincides with the topology of quotient of $\mathcal{O}_{n,b}$. This proves the assertions. \square

Corollary 6.2. *Let $\varphi : B \longrightarrow A$ be a homomorphism of analytic algebras. Then φ is weakly continuous.*

Proof. Suppose that $A = \mathcal{O}_{m,a}/I$, $B = \mathcal{O}_{n,b}/J$ and let $\pi_A : \mathcal{O}_{m,a} \longrightarrow A$, $\pi_B : \mathcal{O}_{n,b} \longrightarrow B$ denote the factor epimorphisms. Then φ lifts to $\tilde{\varphi} : \mathcal{O}_{n,b} \longrightarrow \mathcal{O}_{m,a}$ (see e.g. Grauert-Remmert [GR], Satz II.0.3). Since $\tilde{\varphi}$ is obtained by substitutions of y_i by elements of the maximal ideal of $\mathcal{O}_{n,b}$, it is easy to see that $\tilde{\varphi}$ is weakly continuous. Then the inverse image $(\pi_A \circ \tilde{\varphi})^{-1}(U)$ of an open subset $U \subset A$ is open by Lemma 6.1. This implies that $\varphi^{-1}(U)$ is open again by Lemma 6.1 and that φ is weakly continuous. \square

7. PROJECTOR TO A VECTOR SUBSPACE

Bos and Calvi used the least space of de Boor and Ron to define Taylor projector. A part of their construction can be stated in the following way. Let Z_b be a finite-dimensional vector subspace of the local analytic algebra $\mathcal{O}_{n,b}$. Then there exists a retract (projector) $T_{Z,b} : \mathcal{O}_{n,b} \longrightarrow Z_b$ of vector spaces which is natural to a chosen coordinates $\mathbf{t} := (t_1, \dots, t_n)$. The space Z_b has a structure of Artinian algebra at a general point.

Let $\varphi : \mathcal{O}_{m,a} \longrightarrow \mathcal{O}_{n,b}$ be a homomorphism of analytic algebras. Consider its transposed linear mapping

$${}^t\varphi : \mathbb{C}[\tau] = \mathcal{O}_{n,b}\downarrow \longrightarrow \mathbb{C}[\xi] = \mathcal{O}_{m,a}\downarrow$$

of φ with respect to $B_{m,a}$ and $B_{n,b}$ defined in §5. Since φ is weakly continuous, it satisfies

$$\forall p \in \mathbb{C}[\tau], \forall f \in O_{m,a} : B_{m,a} \langle {}^t\varphi(p) \| f \rangle = B_{m,a} \langle p \| \varphi(f) \rangle$$

and ${}^t\varphi$ is weakly continuous (Bourbaki [Bo₂], 2, §6, n°4). We define the *adjoint*

$${}^s\varphi : \mathbb{C}[\tau] = O_{n,b} \downarrow \longrightarrow \mathbb{C}[\xi] = O_{m,a} \downarrow$$

of φ by ${}^s\varphi := u \circ {}^t\varphi \circ u$, where u denotes the complex conjugation. It is weakly continuous and satisfies

$$\forall p \in \mathbb{C}[\tau], \forall f \in O_{m,a} : S_{m,a} \langle {}^s\varphi(p) | f \rangle := S_{n,b} \langle p | \varphi(f) \rangle.$$

Remark 7.1. The adjoint mapping ${}^s\varphi$ of holomorphic mapping φ of regular analytic algebras is nothing but the push-forward of derivatives of Dirac delta (a distribution with one point support $\{a\}$, see Schwartz [Sc], XXXV). Its concrete forms are given by multivariate versions of Faà di Bruno formula (see e.g. [LP], [Ma]). Note that the image of a homogeneous element by ${}^s\varphi$ is not always homogeneous.

Definition 7.2. For a finite-dimensional vector subspace $Z_b \subset O_{n,b}$, we have defined a non-degenerate sesquilinear form

$$S_{Z,b} : Z_b \downarrow \times Z_b \longrightarrow \mathbb{C}$$

induced from $S_{n,b}$ for some fixed coordinates $\mathbf{t} = (t_1, \dots, t_n)$ (Theorem 5.4). Let $\iota : Z_b \downarrow \longrightarrow O_{n,b} \downarrow$ and $\kappa : Z_b \longrightarrow O_{n,b}$ denote the inclusion mappings. Then we have two adjoint linear mappings $T_{Z,b} := {}^s\iota : O_{n,b} \longrightarrow Z_b$ of ι and ${}^s\kappa : O_{n,b} \longrightarrow Z_b \downarrow$ of κ . Thus we have Diagram 2, where the bold vertical lines imply the sesquilinear pairings.

DIAGRAM 2. Projector

$$\begin{array}{ccc} Z_b & \xrightleftharpoons[\kappa]{\iota} & O_{n,b} = \mathbb{C}[\mathbf{t} - \mathbf{b}] \\ \downarrow S_{Z,b} & T_{Z,b} & \downarrow S_{n,b} \\ Z_b \downarrow & \xrightleftharpoons[{}^s\kappa]{\iota} & O_{n,b} \downarrow = \mathbb{C}[\tau] \end{array}$$

Proposition 7.3. Let $Z_b \subset O_{n,b}$ be a finite-dimensional vector subspace. Then we have the following.

- (1) The projector $T_{Z,b} : O_{n,b} \longrightarrow Z_b$ is weakly continuous.
- (2) We have the equalities $\text{Ker } T_{Z,b} = (Z_b \downarrow)^{\perp_n}$, $(\text{Ker } T_{Z,b})^{\perp_n} = Z_b \downarrow$ where \perp_n denotes the subspace of annihilators with respect to $S_{n,b}$.
- (3) The projector $T_{a,d}$ is a retraction of a vector space, i.e. $T_{Z,b} \circ \kappa : Z_b \longrightarrow Z_b$ is the identity.
- (4) The adjoint ${}^s\kappa$ is a retraction of a vector space, i.e. ${}^s\kappa \circ \iota : Z_b \downarrow \longrightarrow Z_b \downarrow$ is the identity.

Proof. Since the weak topology of $Z_b \downarrow$ with respect to $S_{n,b}$ coincides with the induced one from $\mathbb{C}[\tau]$ by Bourbaki [Bo₂], 4, §1, n°5 Proposition 6, the inclusion ι is weakly continuous. Then its adjoint $T_{Z,b}$ is also so by Bourbaki [Bo₂], 4, §4, n°1, which proves (1). The first equality of

(2) follows from the fact that the sesquilinear form $S_{n,b}$ is non-degenerate. Since $Z_b \downarrow$ is finite-dimensional, it is weakly closed and $Z_b \downarrow = (Z_b \downarrow)^{\perp_n \perp_n}$. Then the first equality implies the second. If $f \in Z_b$ and $p \in Z_b \downarrow$, we have

$$S_{Z,b} \langle p | f \rangle = S_{n,b} \langle \iota(p) | \kappa(f) \rangle = S_{Z,b} \langle p | T_{Z,b} \circ \kappa(f) \rangle.$$

Since $S_{n,b}$ is non-degenerate, this implies (3). It is obvious that (3) implies (4). \square

Remark 7.4. The projector of $T_{Z,b} : O_{n,b} \rightarrow Z_b$ of de Boor-Ron produces a function $T_{Z,b}(f) \in Z_b$, which can take any given set of values for bases of $Z_b \downarrow$ by a suitable choice of f . Therefore $T_{Z,b}$ can be considered as an interpolation by an element of Z_b .

Lemma 7.5. *Let $U \subset \mathbb{C}^n$ be an open subset and $Z \subset O_n(U)$ a vector subspace with $0 < \dim_{\mathbb{C}} Z \leq \infty$. The following conditions (1) and (2) are equivalent at each point $\mathbf{b} \in U$.*

- (1) *The least space $Z_b \downarrow$ is D -invariant.*
- (2) *The kernel $\text{Ker } T_{Z,b} = (Z_b \downarrow)^{\perp_n}$ of $T_{Z,b}$ is a proper ideal.*

The condition (2) is the property of an ideal projector of Birkhoff [Bi] (see de Boor-Shekhtman [BS] for terminology). Equivalence of (1) and (2) appears in M. G. Marinari, H. M. Möller, T. Mora [MMM], Proposition 2.4 and de Boor-Ron [BR₂], Proposition 6.1.

Proof. Suppose that $Z_b \downarrow$ is D -invariant. If $p \in Z_b \downarrow$ and $f \in (Z_b \downarrow)^{\perp_n}$, we have

$$S_{n,b} \langle p | t_i f \rangle = S_{n,b} \langle \partial p / \partial \tau_i | f \rangle = 0 \quad (i = 1, \dots, n)$$

by Lemma 5.6. This implies that $t_i(Z_b \downarrow)^{\perp_n} \subset (Z_b \downarrow)^{\perp_n}$. Then for any $g(t) \in \mathbb{C}[t]$, we have $g(t)(Z_b \downarrow)^{\perp_n} \subset (Z_b \downarrow)^{\perp_n}$. Taking the weak closure, this holds for $g \in O_{n,b}$, which implies that $(Z_b \downarrow)^{\perp_n}$ is an ideal of $O_{n,b}$. This can not be the unit ideal by the retraction property in Proposition 7.3, which completes the proof of (1) \implies (2). In a similar way, we see that, if $(Z_b \downarrow)^{\perp_n}$ is an ideal of $O_{n,a}$, then $(Z_b \downarrow)^{\perp_n \perp_n}$ is D -invariant. Since every homogeneous part of $Z_b \downarrow$ is finite-dimensional and belongs to $Z_b \downarrow$, we see that $Z_b \downarrow$ is weakly closed. Then we have $(Z_b \downarrow)^{\perp_n \perp_n} = Z_b \downarrow$ and (2) \implies (1) follows. \square

Remark 7.6. Under the condition (2) we can endow a structure of a \mathbb{C} -algebra to the vector subspace Z_b such that the linear mapping $T_{Z,b} : O_{n,b} \rightarrow Z_b$ is a factor epimorphism of a \mathbb{C} -algebra (see [BC₂], Corollary 3.6). This algebra has the same dimension as Z_b over \mathbb{C} . If it is finite-dimensional, it is a local analytic algebra of Krull dimension 0. It is Artinian in the sense that it satisfies the descending chain condition of ideals (cf. [Mat]). Thus we have associated to each point of U_Z^{bdl} an Artinian local algebras. Since all the elements of the maximal ideal are nilpotent, Z_b is not a subalgebra of $O_{n,a}$ in general.

Example 7.7. We show a common example. Let us take the vector space

$$Z := \text{Span}(1, s, t, t^2 + st^2, t^3) \subset O_2(\mathbb{R}^2).$$

Its jets are expressed in Diagram 3. For example, the bottom of the diagram implies $j^3(t^3) = (b^3, 0, 3b^2, 0, 3b, 1)$ at (a, b) . Then we have

$$\dim_{\mathbb{C}} R_k(a, b) = \begin{cases} = 1 & (k = 0) \\ = 3 & (k = 1) \\ \leq 5 & (\text{equality} \iff t \neq 0) & (k = 2) \\ = 5 & (\text{everywhere}) & (k \geq 3). \end{cases}$$

DIAGRAM 3. bases of jets of $Z_{(a,b)}\downarrow$

jet space	R_1					
	R_1					
	R_2					
	R_3					
fibre coordinates	$u_{(0,0)}$	$u_{(1,0)}$	$u_{(0,1)}$	$u_{(1,1)}$	$u_{(0,2)}$	$u_{(0,3)}$
1	1	0	0	0	0	0
s	a	1	0	0	0	0
t	b	0	1	0	0	0
$t^2 + st^2$	$b^2 + ab^2$	b^2	$2b(1 + a)$	$2b$	$1 + a$	0
t^3	b^3	0	$3b^2$	0	$3b$	1

Then the set of bundle points of all the jet spaces of Z is the analytically open subset $U_Z^{\text{bdl}} := \{(s, t) : t \neq 0\}$. On the other hand, $Z_{(a,b)}\downarrow$ is D -invariant except $(-1, 0)$. Thus, for example, $(0, 0)$ is not a bundle point of the 2-jet space of Z but $Z_{(0,0)}\downarrow$ is D -invariant.

On the subset U_Z^{bdl} , we see that the least spaces are $Z_{(s,t)}\downarrow = \text{Span}(1, \sigma, \tau, \sigma\tau, \tau^2)$ and the spaces of annihilators of $Z_{(s,t)}\downarrow$ with respect to the sesquilinear form $S_{2,(s,t)}$ are the ideal

$$Z_{(a,b)}\downarrow^{\perp_2} = (s'^2, s't'^2, t'^3) \mathbb{C}\{s', t'\} \quad (s' = s - a, t' = t - b).$$

The factor Artinian algebras by them are all isomorphic on this set. On the thin subset $\{(a, b) : a \neq 0, b = 0\}$, the spaces of annihilators of $Z_{(a,0)}\downarrow$ with respect to $S_{2,(a,0)}$ are the ideals

$$Z_{(a,0)}\downarrow^{\perp_2} = (s'^2, s't', t'^4) \mathbb{C}\{s', t'\} \quad (s' = s - a, t' = t).$$

The factor Artinian algebras by them are different from those on U_Z^{bdl} . We will see this example again in Example 11.4.

8. INTRINSIC TREATMENT OF THE LEAST SPACE

Here we show intrinsic treatment of the least space, D -invariance property and introduce the Artinian algebra associated to a D -invariant point.

Let M be an n -dimensional complex manifold and φ a local parametrisation around $\mathbf{b} \in M$ with $\varphi(0) = \mathbf{b}$. Let s be the local coordinates for φ , then $(f \circ \Phi)_0\downarrow$ is expressed as a homogeneous polynomial in σ . By the definition of the least part, if ψ is another local parametrisation of M around \mathbf{b} with local coordinates t , we have

$$(f \circ \Psi)_0\downarrow(\tau) = (f \circ \Phi)_0\downarrow(J_\theta \sigma),$$

where $J_\theta := \partial(t)/\partial(s)$ denotes the Jacobian matrix evaluated at $\mathbf{0}$. Let us define the least part $f_b\downarrow$ of f as the data

$$\{(f \circ \Phi)_0\downarrow : \Phi \text{ is a local parametrisation of } M \text{ at } \mathbf{b}\}$$

related by the equality above. Then the least part $(f \circ \Phi)_0\downarrow$ is a coordinate expression of $f_b\downarrow$. The k -th homogeneous part of $\mathbb{C}[\tau]$ can be seen as a coordinate expression of the evaluation at

\mathbf{b} of the k -fold symmetric tensor product over \mathcal{O}_U of the usual cotangent space \mathcal{T}^* of \mathbb{C}^n . Thus we have the intrinsic form of $\mathbb{C}[\tau]$ (cf. Quillen [Qu], p.2)

$$\mathcal{O}_{M,b}\downarrow := \{f_b\downarrow : f \in \mathcal{O}_{M,b}\} \cong \bigoplus_k \mathcal{T}^*(\mathbf{b})^{\odot k},$$

where the tensor product is taken over \mathbb{C} . Let $Z \subset \mathcal{O}_{M,b}$ be a finite-dimensional vector subspace and let $Z_b^\varphi\downarrow$ denote the least space of $\{f \circ \Phi : f \in Z\}$ with respect to s . Let us define the least space $Z_b\downarrow \subset \mathcal{O}_{M,b}\downarrow$ at $\mathbf{0}$ by

$$Z_b\downarrow := \{f_b\downarrow : f \in Z\} \subset \mathcal{O}_{M,b}\downarrow.$$

The property that \mathbf{a} is a bundle points of all jet spaces of $Z_b^\varphi\downarrow$ is independent of φ by Corollary 3.7. Let M_Z^{bdl} denote the set of bundle points \mathbf{b} of all the jet spaces of $Z_b^\varphi\downarrow$.

Theorem 8.1. *Let M be an n -dimensional complex manifold and $Z \subset \mathcal{O}(M)$ be a finite-dimensional vector subspace. Take two local parametrisations around $\mathbf{b} \in M$ which induce isomorphisms $\varphi : \mathcal{O}_{n,b} \rightarrow \mathbb{C}\{s\}$ and $\psi : \mathcal{O}_{n,b} \rightarrow \mathbb{C}\{t\}$ such that $\varphi = \theta \circ \psi$ for the third isomorphism θ of algebras. Then we have the following.*

(1) *Let*

$$\tilde{j}_\theta : \mathbb{C}[\tau] \rightarrow \mathbb{C}[\sigma]$$

denote the isomorphism of the algebras defined by substitution of τ by $J_\theta\sigma$ then $\tilde{j}_\theta(Z_b^\varphi\downarrow) = Z_b^\psi\downarrow$, which implies that $Z_b\downarrow \subset \mathcal{O}_{M,b}\downarrow = \bigoplus_k \mathcal{T}^(\mathbf{b})^{\odot k}$.*

- (2) *The space $Z_b^\varphi\downarrow$ is D -invariant if and only if $Z_b^\psi\downarrow$ is so, namely D -invariance property of $Z_b\downarrow$ is well-defined. If M_Z^{inv} denotes the set of D -invariant points we have $M_Z^{\text{bdl}} \subset M_Z^{\text{inv}}$.*
- (3) *If $\mathbf{b} \in M_Z^{\text{inv}}$, the vector subspace $Z_b \subset \mathcal{O}_b$ has a structure of a local Artinian algebra such that the projector $T_{Z,b} : \mathcal{O}_b \rightarrow Z_b$ is a factor homomorphism. The adjoint linear isomorphism*

$${}^s\tilde{j}_\theta : \mathbb{C}\{s\} \rightarrow \mathbb{C}\{t\}$$

of \tilde{j}_θ is also an isomorphism of analytic algebras. It induces an isomorphism

$$\mathbb{C}\{s\}/(Z_b^\psi\downarrow)^{\perp n} \rightarrow \mathbb{C}\{t\}/(Z_b^\varphi\downarrow)^{\perp n}$$

of algebras. In other words, Z_b has a canonical structure of an Artinian algebra, symbolically expressed as $\mathcal{O}_{n,b}/(Z_b\downarrow)^{\perp n}$.

The homomorphism ${}^s\tilde{j}_\theta$ is nothing but the homomorphism corresponding to the transformation defined by the linear part of θ .

Proof. The assertions of (1) are already explained above. Let us consider the following implications:

$$\begin{aligned} f \in (Z_b^\psi\downarrow)^{\perp n} &\iff f \in (\tilde{j}_\theta(Z_b^\varphi\downarrow))^{\perp n} \iff \forall p \in \tilde{j}_\theta(Z_b^\varphi\downarrow) : S_{n,\mathbf{0}} \langle p | f \rangle = 0 \\ &\iff \forall q \in Z_b^\varphi\downarrow : S_{n,\mathbf{0}} \langle \tilde{j}_\theta(q) | f \rangle = 0 \iff \forall q \in Z_b^\varphi\downarrow : S_{n,\mathbf{0}} \langle q | {}^s\tilde{j}_\theta(f) \rangle = 0 \\ &\iff {}^s\tilde{j}_\theta(f) \in (Z_b^\varphi\downarrow)^{\perp n} \iff f \in ({}^s\tilde{j}_\theta)^{-1}((Z_b^\varphi\downarrow)^{\perp n}). \end{aligned}$$

This proves

$$(Z_b^\psi\downarrow)^{\perp n} = ({}^s\tilde{j}_\theta)^{-1}((Z_b^\varphi\downarrow)^{\perp n})$$

and $(Z_b^\varphi\downarrow)^{\perp n}$ is the image of $(Z_b^\psi\downarrow)^{\perp n}$ by the linear isomorphism ${}^s\tilde{j}_\theta$. This isomorphism is obtained by substituting s by $J_\theta t$ and hence it is even an algebra isomorphism. Then $(Z_b^\varphi\downarrow)^{\perp n}$ is an ideal

if and only if $(Z_b^\psi \downarrow)^{\perp_n}$ is so and hence $Z_b^\varphi \downarrow$ is D -invariant if and only if $Z_b^\psi \downarrow$ is so. When this is the case, $\mathbb{C}\{s\}/(Z_b^\varphi \downarrow)^{\perp_n}$ and $\mathbb{C}\{t\}/(Z_b^\psi \downarrow)^{\perp_n}$ are isomorphic as algebras. The assertion on the relation with bundle points follows from Theorem 4.2. \square

Theorem 8.2. *Let $\varphi : O_{m,a} := \mathbb{C}\{x\} \longrightarrow O_{n,b} := \mathbb{C}\{t\}$ be a homomorphism of analytic algebras and let ${}^s\varphi : \mathbb{C}[\tau] \longrightarrow \mathbb{C}[\xi]$ denote its adjoint mapping with respect to the standard sesquilinear forms $S_{m,a}$ and $S_{n,b}$. If $Q \subset \mathbb{C}[\tau]$ is a finite-dimensional vector subspace, φ induces a monomorphism ψ of the factor vector spaces:*

$$\psi : O_{m,a}/({}^s\varphi(Q))^{\perp_m} \longrightarrow O_{n,b}/Q^{\perp_n}.$$

Furthermore we have the following.

- (1) *If φ is an epimorphism, ψ is an isomorphism.*
- (2) *If Q is D -invariant, so is ${}^s\varphi(Q)$ and ψ is a homomorphism of Artinian algebras.*
- (3) *If φ is an epimorphism and Q is D -invariant, then ψ is an isomorphism of Artinian algebras.*

Proof. To prove that ψ exists and it is a monomorphism, we have only to prove the equality $({}^s\varphi(Q))^{\perp_m} = \varphi^{-1}(Q^{\perp_n})$. Since

$$f \in ({}^s\varphi(Q))^{\perp_m} \iff \forall q \in Q : S_{m,a} \langle {}^s\varphi(q) | f \rangle = 0,$$

$$f \in \varphi^{-1}(Q^{\perp_n}) \iff \forall q \in Q : S_{n,b} \langle q | \varphi(f) \rangle = 0,$$

the equality $S_{m,a} \langle {}^s\varphi(q) | f \rangle = S_{n,b} \langle q | \varphi(f) \rangle$ implies $({}^s\varphi(Q))^{\perp_m} = \varphi^{-1}(Q^{\perp_n})$.

If φ is surjective, it is obvious that ψ is bijective. If Q is D -invariant, Q^{\perp_n} is a proper ideal by Theorem 7.5. Then $({}^s\varphi(Q))^{\perp_m} = \varphi^{-1}(Q^{\perp_n})$ is also so and ${}^s\varphi(Q)$ is D -invariant again by Theorem 7.5. The assertion ψ is an algebra homomorphism is now obvious. The algebra $O_{n,b}/Q^{\perp_n}$ is Artinian because $\dim_{\mathbb{C}} O_{n,b}/Q^{\perp_n} = \dim_{\mathbb{C}} Q < \infty$ by [Bo2], 4, §1, n°5 Proposition 5 and Corollary 5.1. The assertion (3) follows from (1) and (2). \square

9. HIGHER ORDER TANGENTS OF BOS AND CALVI

Following the method of Bos-Calvi, we introduce a set $D_{X,a}^{\varphi,d}$ of higher order tangents of complex analytic submanifold X of an open subset $\Omega \subset \mathbb{C}^m$ at $a \in X$. It is not an intrinsic object associated to X_a as a germ of a complex space but it reflects the properties of the embedding germ $X_a \subset \mathbb{C}_a^m$. It is also dependent upon the choice of the local parametrisation φ of X_a in general. It is a dual space of the space $P^d(X_a)$ of the polynomial functions of degrees at most d . We will often skip the adjective “higher order” for tangents.

Let X_a be the germ of a regular complex submanifold X of an open subset $\Omega \subset \mathbb{C}^m$ at a . The algebra $O_{X,a}$ of germs of holomorphic functions on a neighbourhood (in X) of a at a is the factor algebra of $O_{m,a} = \mathbb{C}\{x\}$ ($x := (x_1, \dots, x_m)$) by the ideal I_a of convergent power series vanishing on the germ X_a : $O_{X,a} \cong O_{m,a}/I_a$. Hence $O_{X,a}$ is an analytic local algebra. Let $\pi : O_{m,a} \longrightarrow O_{X,a}$ denotes the factor epimorphism. By the assumption that X is a submanifold, there is an isomorphism

$$\psi : O_{X,a} \longrightarrow O_{n,b} = \mathbb{C}\{t - b\} \quad (t := (t_1, \dots, t_n), \dim X = n).$$

Then we have the epimorphism

$$\varphi := \psi \circ \pi : O_{m,a} \longrightarrow O_{n,b}.$$

This is just the epimorphism defined by the pullback by the germ of embedding

$$\Phi = (\Phi_1, \dots, \Phi_m) : \mathbb{C}_b^n \longrightarrow \mathbb{C}_a^m,$$

namely $\varphi(f) = f \circ \Phi$. We call this φ or Φ a *local parametrisation* of X at \mathbf{a} . Let $\mathbb{C}[\Phi] \subset \mathcal{O}_{n,b}$ denote the algebras of pullbacks of $\mathbb{C}[\mathbf{x}]$ by φ . Let

$$P(X_a) = \mathbb{C}[\mathbf{x}]|_{X_a} \subset \mathcal{O}_{X,a}$$

denote the ring of germs of polynomial function on X_a . It is easy to see the following.

Lemma 9.1. *We have the algebra isomorphism*

$$\mathbb{C}[\Phi] := \varphi(\mathbb{C}[\mathbf{x}]) = \psi(P(X_a)) \cong P(X_a).$$

By the general property of the transposed mapping of a surjective homomorphism, ${}^t\varphi$ is injective and hence ${}^s\varphi$ is also so. The image $D_{X,a}^\varphi := {}^s\varphi(\mathbb{C}[\tau])$ is geometrically the space of higher order tangents of X at \mathbf{a} because of the property $I_a^{\perp m} = D_{X,a}^\varphi$ shown below. We call its elements ${}^s\varphi(p)$ the *Bos-Calvi (higher order) tangents*. The sesquilinear form $S_{n,b}$ induces a non-degenerate sesquilinear form

$$S_{X,a}^\varphi : {}^s\varphi(\mathbb{C}[\tau]) \times \mathcal{O}_{X,a} \longrightarrow \mathbb{C}$$

through ψ and ${}^s\varphi$ (see Diagram 5 in §10). Let $I_a^{\perp m}$ and $I_a^{\perp X}$ denote the space of annihilators of I_a with respect to $S_{m,a}^\varphi$ and $S_{X,a}^\varphi$ respectively.

Proposition 9.2. *We have $(D_{X,a}^\varphi)^{\perp m} = I_a$ and $I_a^{\perp m} = D_{X,a}^\varphi$. Hence $D_{X,a}^\varphi$ is independent of the local parametrisation φ .*

Thus we may omit the superscript φ of $D_{X,a}^\varphi$.

Proof. The first equality follows from the following implications.

$$\begin{aligned} f \in I_a &\iff \varphi(f) = 0 \iff \forall p \in \mathbb{C}[\tau] : S_{n,b} \langle p | \varphi(f) \rangle = 0 \\ &\iff \forall p \in \mathbb{C}[\tau] : S_{m,a} \langle {}^s\varphi(p) | f \rangle = 0 \iff f \in ({}^s\varphi(\mathbb{C}[\tau]))^{\perp m} \iff f \in (D_{X,a}^\varphi)^{\perp m} \end{aligned}$$

Then we have $I_a^{\perp m} = (D_{X,a}^\varphi)^{\perp m \perp m}$. To see the equality $I_a^{\perp m} = D_{X,a}^\varphi$, we have only to prove that $(D_{X,a}^\varphi)^{\perp m \perp m} = D_{X,a}^\varphi$ or that $D_{X,a}^\varphi = {}^s\varphi(\mathbb{C}[\tau])$ is weakly closed in $\mathbb{C}[\xi]$. Since φ is an open continuous epimorphism by Lemma 6.1, ${}^t\varphi(\mathbb{C}[\tau])$ is weakly closed by Bourbaki [Bo₂], 4, §4, n°1 Proposition 4. Since $u(\mathbb{C}[\tau]) = \mathbb{C}[\tau]$ and $u : \mathbb{C}[\xi] \longrightarrow \mathbb{C}[\xi]$ is a homeomorphism, ${}^s\varphi(\mathbb{C}[\tau])$ is also weakly closed. \square

Let

$$P^d(X_a) := \{p \bmod I_a : p \in \mathbb{C}[\mathbf{x}], \deg p \leq d\} \subset P(X_a) \subset \mathcal{O}_{X,a}$$

denote the vector space of *polynomial functions* on X at \mathbf{a} of degree at most d .

Remark 9.3. If X_a is defined by an ideal $I_a \subset \mathcal{O}_{m,a}$, the algebra $\mathbb{C}[\mathbf{x}]/(I_a + \mathfrak{m}_a^{d+1}) \cap \mathbb{C}[\mathbf{x}]$ is different from the vector space $P^d(X_a)$. The canonical mapping

$$\pi_d : P^d(X_a) \longrightarrow \mathbb{C}[\mathbf{x}]/(I_a + \mathfrak{m}_a^{d+1}) \cap \mathbb{C}[\mathbf{x}]$$

is surjective but not always injective.

Let

$$\mathbb{C}[\mathbf{x}]^d := \{f(\mathbf{x}) : f \in \mathbb{C}[\mathbf{x}], \deg f \leq d\} \subset \mathbb{C}[\mathbf{x}]$$

denote the vector space of polynomials of degrees at most d . If we put

$$\mathbb{C}[\Phi]^d := \varphi(\mathbb{C}[\mathbf{x}]^d) = \psi(P^d(X_a)) \subset O_{n,b}$$

using a local parametrisation Φ , we have an increasing sequence of finite-dimensional vector subspaces

$$\mathbb{C} = \mathbb{C}[\Phi]^0 \subset \mathbb{C}[\Phi]^1 \subset \dots$$

of the \mathbb{C} -algebra $\mathbb{C}[\Phi] \subset O_{n,b}$. Then we have a sequence

$$\mathbb{C} = \mathbb{C}[\Phi]_b^0 \downarrow \subset \mathbb{C}[\Phi]_b^1 \downarrow \subset \dots$$

of finite-dimensional vector subspaces of $\mathbb{C}[\tau] = O_{n,b} \downarrow$. Let us fix the degree d hereafter. Recall that $S_{n,b}$ induces a non-degenerate sesquilinear form

$$S_{n,b}^{\varphi,d} : \mathbb{C}[\Phi]_b^d \downarrow \times \mathbb{C}[\Phi]^d \longrightarrow \mathbb{C}$$

by Theorem 5.4.

Definition 9.4. Let us put

$$D_{X,a}^{\varphi,d} := {}^s\varphi(\mathbb{C}[\Phi]_b^d \downarrow).$$

Since there is a natural isomorphism $\psi|_{P^d(X_a)} : P^d(X_a) \longrightarrow \mathbb{C}[\Phi]^d$ by Lemma 9.1, we see that $S_{n,b}^{\varphi,d}$ induces a non-degenerate sesquilinear form

$$S_{X,a}^{\varphi,d} : D_{X,a}^{\varphi,d} \times P^d(X_a) \longrightarrow \mathbb{C}.$$

We call the elements of $D_{X,a}^{\varphi,d}$ *Bos-Calvi tangents of X_a of dual degree d* .

If $m > n$, some element of $D_{X,a}^{\varphi,d}$ has a degree higher than d as we will see in the examples below. This is a simple consequence of Dirichlet's box principle.

Lemma 9.5. *Let X_a be the germ of a regular complex submanifold X of an open subset $\Omega \subset \mathbb{C}^m$ at \mathbf{a} . Take two local parametrisations*

$$\Phi : \mathbb{C}_b^n \longrightarrow \mathbb{C}_a^m, \quad \Psi : \mathbb{C}_{b'}^n \longrightarrow \mathbb{C}_a^m$$

of X_a . Let $\Theta : \mathbb{C}_b^n \longrightarrow \mathbb{C}_{b'}^n$ denote the biholomorphic germ (coordinate change) such that $\Phi = \Psi \circ \Theta$ and $\varphi = \theta \circ \psi$. Then we have the following.

- (1) *The isomorphism θ induces that of $\mathbb{C}[\Psi]^d$ and $\mathbb{C}[\Phi]^d$.*
- (2) *If \mathbf{b} is a bundle points of all the jet spaces of $\mathbb{C}[\Phi]^d$, it is the same for $\mathbb{C}[\Psi]^d$.*
- (3) *If the inclusion*

$$(*) \quad {}^s\theta|_{\mathbb{C}[\Phi]^d \downarrow}(\mathbb{C}[\Phi]_b^d \downarrow) \subset \mathbb{C}[\Psi]_b^d \downarrow$$

holds, then we have $D_{X,a}^{\varphi,d} = D_{X,a}^{\psi,d}$.

Proof. The isomorphism θ associated to Θ restricts to an isomorphism

$$\theta|_{\mathbb{C}[\Psi]^d} : \mathbb{C}[\Psi]^d \longrightarrow \mathbb{C}[\Phi]^d$$

through the isomorphisms to the subspace $P^d(X) \subset O_{X,a}$ (by Lemma 9.1). As stated after Definition 3.6, the bundle property is preserved by a coordinate transformation.

To prove (3), take the adjoint isomorphism

$${}^s(\theta|_{\mathbb{C}[\Psi]^d}) : \mathbb{C}[\Phi]_{b\downarrow}^d \longrightarrow \mathbb{C}[\Psi]_{b\downarrow}^d$$

of vector spaces. Let

$$\iota : \mathbb{C}[\Phi]^d \hookrightarrow \mathbb{C}\{\mathbf{t}\}, \quad \iota' : \mathbb{C}[\Psi]^d \hookrightarrow \mathbb{C}\{\mathbf{t}'\}$$

denote the inclusions. Since the projectors ${}^s\iota$ and ${}^s\iota'$ are retractions (Proposition 7.3, (4)), the

DIAGRAM 4. Coordinate transformation

$$\begin{array}{ccccc}
 & & \text{id} & & \\
 & \swarrow & \text{---} & \searrow & \\
 \mathbb{C}[\Phi]_{b\downarrow}^d & \xrightarrow{\iota} & \mathbb{C}[\tau] & \xrightarrow{{}^s\iota} & \mathbb{C}[\Phi]_{b\downarrow}^d \\
 \downarrow ({}^s\theta)|_{\mathbb{C}[\Phi]_{b\downarrow}^d} & & \downarrow {}^s\theta & & \downarrow {}^s(\theta|_{\mathbb{C}[\Psi]^d}) \\
 \mathbb{C}[\Psi]_{b\downarrow}^d & \xrightarrow{\iota'} & \mathbb{C}[\tau'] & \xrightarrow{{}^s\iota'} & \mathbb{C}[\Psi]_{b\downarrow}^d \\
 & \nwarrow & \text{---} & \swarrow & \\
 & & \text{id} & &
 \end{array}$$

compositions $\mathbb{C}[\Phi]_{b\downarrow}^d \xrightarrow{\iota} \mathbb{C}[\tau] \xrightarrow{{}^s\iota} \mathbb{C}[\Phi]_{b\downarrow}^d$ and $\mathbb{C}[\Psi]_{b\downarrow}^d \xrightarrow{\iota'} \mathbb{C}[\tau'] \xrightarrow{{}^s\iota'} \mathbb{C}[\Psi]_{b\downarrow}^d$ are the identities. Thus we have Diagram 5 (the dash arrow is our special assumption (*)), whose commutativity implies that ${}^s(\theta|_{\mathbb{C}[\Psi]^d}) = ({}^s\theta)|_{\mathbb{C}[\Phi]_{b\downarrow}^d}$. It follows that

$$\begin{aligned}
 D_{X,a}^{\psi,d} &= {}^s\psi(\mathbb{C}[\Psi]_{b\downarrow}^d) = {}^s\psi \circ {}^s(\theta|_{\mathbb{C}[\Psi]^d})(\mathbb{C}[\Phi]_{b\downarrow}^d) \\
 &= {}^s\psi \circ ({}^s\theta)|_{\mathbb{C}[\Phi]_{b\downarrow}^d}(\mathbb{C}[\Phi]_{b\downarrow}^d) = {}^s\varphi(\mathbb{C}[\Phi]_{b\downarrow}^d) = D_{X,a}^{\varphi,d}.
 \end{aligned}$$

□

Example 9.6. If $\dim X \geq 2$, $D_{X,a}^{\varphi,d}$ is very sensitive to a change of the local parametrisation. Let us consider the surface $X \subset \mathbb{C}^3$ defined by $x_3 = x_2^2$. Take two global parametrisations:

$$\varphi : x_1 = s_1, x_2 = s_2, x_3 = s_2^2;$$

$$\psi : x_1 = t_1 + t_2, x_2 = t_2, x_3 = t_2^2.$$

This is a simple linear transformation of local coordinates: $t_1 = s_1 + s_2, t_2 = s_2$. The dimensions of the spaces $R_k(\mathbf{b})$ for $\mathbb{C}[\Phi]^1$ defined in §3 are constant:

$$\dim_{\mathbb{C}} R_0(\mathbf{b}) = 1, \dim_{\mathbb{C}} R_1(\mathbf{b}) = 3, \dim_{\mathbb{C}} R_k(\mathbf{b}) = 4 \ (k \geq 2)$$

for any $\mathbf{b} \in \mathbb{R}^2$. This means that all points are bundle points of all the jet spaces of $\mathbb{C}[\Phi]^1$. Then $\mathbb{C}[\Psi]^1$ has also the bundle property everywhere by Lemma 9.5. Let us put $\sigma_i := s_{i,b\downarrow}, \tau_i := t_{i,b\downarrow}, \xi_i := x_{i,a\downarrow}$ and $\eta_i := y_{i,a\downarrow}$. The least spaces of the pullbacks of polynomials of degree at most 1 with respect to them are

$$\mathbb{C}[\Phi]_{b\downarrow}^1 = \text{Span}(1, \sigma_1, \sigma_2, \sigma_2^2), \quad \mathbb{C}[\Psi]_{b\downarrow}^1 = \text{Span}(1, \tau_1, \tau_2, \tau_2^2),$$

and they are D -invariant (which is also a consequence of Theorem 4.2) everywhere. The push-forwards of them are:

$$\begin{aligned} {}^s\varphi(\sigma_1) &= \xi_1, & {}^s\varphi(\sigma_2) &= \xi_2 + 2s_2\xi_3 \\ {}^s\varphi(\sigma_2^2) &= 2\xi_3 + \xi_2^2 + 4s_2\xi_2\xi_3 + 4s_2^2\xi_3^2; \\ {}^s\psi(\tau_1) &= \xi_1, & {}^s\psi(\tau_2) &= \xi_1 + \xi_2 + 2t_2\xi_3, \\ {}^s\psi(\tau_2^2) &= 2\xi_3 + \xi_1^2 + 2\xi_1\xi_2 + 4t_2\xi_1\xi_3 + \xi_2^2 + 4t_2\xi_2\xi_3 + 4t_2^2\xi_3^2. \end{aligned}$$

The monomial ξ_1^2 appears in the linear span of the latter but not in that of the former. Hence two sets of Bos-Calvi tangents $D_{X,a}^{\varphi,1}$ and $D_{X,a}^{\psi,1}$ at

$$\mathbf{a} = \Phi(s_1, s_2) = \Psi(t_1, t_2) \in X$$

are different.

10. TAYLOR PROJECTOR

Now we can introduce the Taylor projector of degree d using Bos-Calvi tangents defined in the previous section. In general, this projector depends upon the local parametrisation.

Assume the same as in the previous section for $\mathbf{a} \in X \subset \mathbb{C}^m$ and its local parametrisation $\varphi : \mathcal{O}_{m,a} \longrightarrow \mathcal{O}_{n,b}$.

Definition 10.1. Let $\iota : D_{X,a}^{\varphi,d} \longrightarrow D_{X,a}$ denote the inclusion mapping. We call its adjoint linear mapping

$$T_a^{\varphi,d} := {}^s\iota : \mathcal{O}_{X,a} \longrightarrow P^d(X_a)$$

φ -Taylor projector of degree d at \mathbf{a} . This was first introduced by Bos and Calvi [BC₁], [BC₂]. It is a little different from ours. Their projector is the composition of our $T_a^{\varphi,d}$ and the factor epimorphism $\mathcal{O}_{m,a} \longrightarrow \mathcal{O}_{X,a}$. The image $T_a^{\varphi,d}(f)$ is called the φ -Taylor polynomial of degree d .

We know the following by Proposition 7.3.

- (1) The φ -Taylor projector $T_a^{\varphi,d}$ is weakly continuous.
- (2) We have the equalities

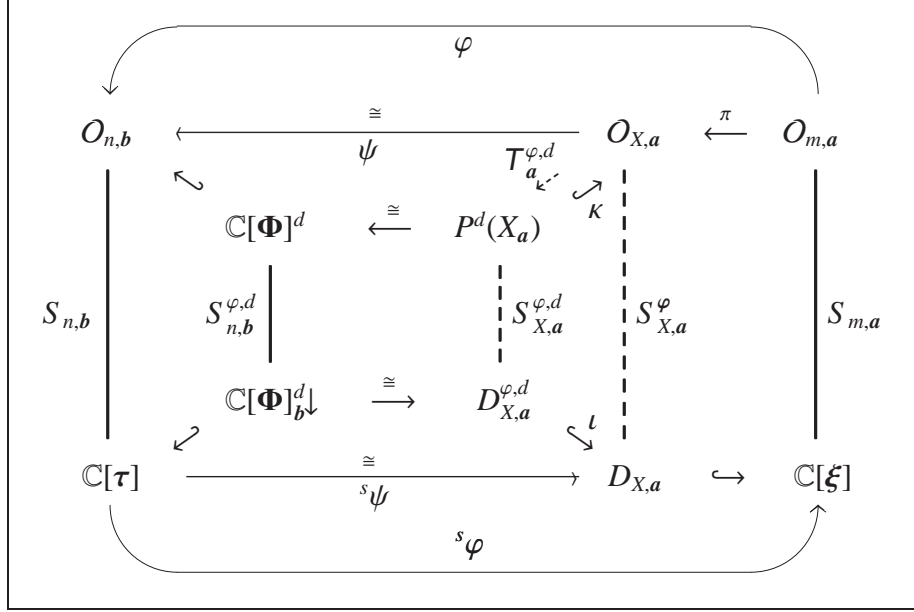
$$\text{Ker } T_a^{\varphi,d} = (D_{X,a}^{\varphi,d})^{\perp_X}, \quad (\text{Ker } T_a^{\varphi,d})^{\perp_X} = D_{X,a}^{\varphi,d},$$

$$\text{Ker } T_a^{\varphi,d} \circ \pi = (D_{X,a}^{\varphi,d})^{\perp_m}, \quad (\text{Ker } T_a^{\varphi,d} \circ \pi)^{\perp_m} = D_{X,a}^{\varphi,d},$$

where \perp_X (resp. \perp_m) denotes the space of annihilators with respect to $S_{X,a}^{\varphi}$ (resp. $S_{m,a}^{\varphi}$).

- (3) The φ -Taylor projector $T_a^{\varphi,d}$ is a retraction of a vector space, i.e. $T_a^{\varphi,d} \circ \kappa : P^d(X_a) \longrightarrow P^d(X_a)$ is the identity, where $\kappa : P^d(X_a) \longrightarrow \mathcal{O}_{X,a}$ denotes the inclusion.

Summing up we have Diagram 5, where the bold lines imply the dual pairings $S_{m,a}$ and $S_{n,b}$ with respect to the affine coordinates \mathbf{x} and \mathbf{t} and the dotted ones imply those induced by $S_{n,b}$ through isomorphisms ψ and ${}^s\psi$. The upper half and the lower half correspond mutually by taking adjoint, excepting the inclusions in the upper half. Let us recall that, even in 1-dimensional case, there exists a point with two different Taylor projectors as follows.

DIAGRAM 5. Dualities by the sesquilinear forms \downarrow 

Example 10.2 (Bos-Calvi [BC₂], Example 4.2). Let $X \subset \mathbb{C}^2$ be the plane curve defined by $y - x^2 - x^6 = 0$. Take two local parametrisations of X at $\mathbf{0}$:

$$\begin{aligned} \varphi: \mathbb{C}_0 &\longrightarrow \mathbb{C}_0^2, & s &\longmapsto \Phi(s) := (s, s^2 + s^6), \\ \psi: \mathbb{C}_0 &\longrightarrow \mathbb{C}_0^2, & t &\longmapsto \Psi(t) := (t + t^2, (t + t^2)^2 + (t + t^2)^6). \end{aligned}$$

Then, putting $\sigma := s_b \downarrow$, $\tau := t_b \downarrow$, we have

$$\mathbb{C}[\Phi]_b^2 \downarrow = \text{Span}(1, \sigma, \sigma^2, \sigma^3, \sigma^4, \sigma^6), \quad \mathbb{C}[\Psi]_b^2 \downarrow = \text{Span}(1, \tau, \tau^2, \tau^3, \tau^4, \tau^6).$$

These lack the degree 5 term and are not D -invariant. We can confirm that

$$T_0^{\varphi,2}(x^5) = 0, \quad T_0^{\psi,2}(x^5) = 5(y - x^2).$$

This means that two different local parametrisation sometimes yield different Taylor projectors and that our Taylor projector does not necessarily coincide with the ordinary Taylor projector.

Definition 10.3. Let us put

$$\begin{aligned} \lambda_\Phi(d) &:= \max \left\{ k : \bigoplus_{i=0}^k \frac{\mathfrak{m}_{n,b}^i}{\mathfrak{m}_{n,b}^{i+1}} \subset \mathbb{C}[\Phi]_b^d \downarrow \right\} = \max \left\{ k : \mathbb{C}[\tau]^k \subset \mathbb{C}[\Phi]_b^d \downarrow \right\} \\ &= \max \left\{ k : (\mathbb{C}[\Phi]_b^d \downarrow)^{\perp_n} \subset \mathfrak{m}_{n,b}^{k+1} \right\} = \max \left\{ k : O_{X,a} \subset P^d(X_a) + \mathfrak{m}_{X,a}^{k+1} \right\}. \end{aligned}$$

This is independent of the parametrisation Φ .

Proposition 10.4. We have $\lambda_\Phi(d) \geq d$. Hence $\mathbb{C}[\Phi]_b^d \downarrow \cap \bigoplus_{i=0}^{d+1} \mathfrak{m}_{n,b}^i / \mathfrak{m}_{n,b}^{i+1}$ is a D -invariant subspace.

Proof. Since Φ generates $\mathfrak{m}_{n,b}$, all t_1, \dots, t_n appear in the linear terms of $\mathbb{C}[\Phi]^1$. Hence all terms of degree $k \leq d$ in τ_1, \dots, τ_n appear in $\mathbb{C}[\Phi]_b^d \downarrow$. \square

By the inequality $\lambda_\Phi(d) \geq d$, we have the following formal error bound for our φ -Taylor polynomial $T_a^{\varphi,d}(f)$:

$$f - \kappa \circ T_a^{\varphi,d}(f) \in (D_{X,a}^{\varphi,d})^{\perp_X} \subset \mathfrak{m}_{X,a}^{\lambda_\Phi(d)+1} \quad (f \in \mathcal{O}_{X,a}),$$

where $\kappa : P^d(X_a) \longrightarrow \mathcal{O}_{X,a}$ denotes the inclusion. This formal error bound is equal to or smaller than that of the ordinary Taylor polynomial $T_a^{*d}(f)$:

$$f - \kappa(T_a^{*d}(f) \bmod I_a) \in \mathfrak{m}_{X,a}^{d+1} \quad (f \in \mathcal{O}_{X,a}).$$

The author does not know whether

$$\text{ord}_{X,a}(f - \kappa \circ T_a^{\varphi,d}(f)) \geq \text{ord}_{X,a}(f - \kappa(T_a^{*d}(f) \bmod I_a))$$

holds or not.

11. D -INVARIANCE AND THE ASSOCIATED ARTINIAN ALGEBRA

In the case of plane algebraic curves, Bos-Calvi [BC₁], [BC₂] proved that the Taylor projector of degree d is well-defined at a general point. We cannot do so in a higher-dimensional manifold X embedded in \mathbb{C}^m . We prove a little weaker assertion for this general case: there exists a unique structure of Artinian algebra up to isomorphism on $P^d(X_a)$ for a general point of $a \in X$ such that the φ -Taylor projector is an algebra homomorphism.

Theorem 11.1. *The property of D -invariance of $\mathbb{C}[\Phi]_b^d \downarrow$ is independent of the choice of the local parametrisation φ .*

Proof. Take two local parametrisations φ and ψ of X_a with $\varphi = \theta \circ \psi$, where θ corresponds to a biholomorphic germ (coordinate transformation) around $b \in \mathbb{C}^n$. Take care of the fact that the least spaces $\mathbb{C}[\Phi]_b^d \downarrow$ and $\mathbb{C}[\Psi]_b^d \downarrow$ are contravariant objects though they are treated as covariant objects in the definition of Bos-Calvi tangents. Hence they are not connected by the Faà di Bruno type formula for Θ directly here.

By the definition of the least operator in §2, θ induces an algebra isomorphism \tilde{j}_θ of $\mathbb{C}[\tau]$ determined only by the Jacobian matrix of J_θ of θ evaluated at b . Explicitly, \tilde{j}_θ is the algebra isomorphism obtained by substituting τ by $J_\theta \tau$, a linear transformation of variables in the polynomial algebra. A linear transformation of variables preserves the property of D -invariance of a set of polynomials. \square

Definition 11.2. If $\mathbb{C}[\Phi]_b^d \downarrow$ is D -invariant for all (or some) local parametrisation φ , we call $a = \Phi(b) \in X$ *D -invariant of degree d* . If a is D -invariant of degree d for all $d \in \mathbb{N}$, we call it *D -invariant of degree ∞* . If the φ -Taylor projector of degree d at $a \in X$ is independent of the local parametrisation, we call a *Taylorian of degree d* following Bos and Calvi. If a is Taylorian of degree d for all $d \in \mathbb{N}$, we call it *Taylorian of degree ∞* .

Proposition 11.3. *Let X be a regular complex submanifold of an open subset of \mathbb{C}^m with $\dim X \geq 1$. For a point $a \in X$, the following conditions (1), ..., (4) are equivalent for $d \in \mathbb{N}$.*

- (1) a is D -invariant of degree d .
- (2) $(\mathbb{C}[\Phi]_b^d \downarrow)^{\perp_n}$ is an ideal of $\mathcal{O}_{n,b}$ for all (or some) local parametrisation φ .

- (3) $\text{Ker } T_a^{\varphi,d} = (D_{x,a}^{\varphi,d})^{\perp_X}$ is an ideal of $\mathcal{O}_{X,a}$ for all (or some) φ for all (or some) φ .
- (4) $D_{x,a}^{\varphi,d}$ is D -invariant in $\mathbb{C}[\xi]$ for all (or some) local parametrisation φ .

If \mathbf{b} is a bundle point of all the jet spaces of $\mathbb{C}[\Phi]^d$, all of these conditions hold.

Proof. Note that the condition of bundle point is independent of the local parametrisation (Lemma 9.5). The equivalence (1) \iff (2) is proved in Theorem 8.1. Since an image or an inverse image of an ideal by a ring epimorphism is an ideal, the equivalences (2) \iff (3) holds. The equivalence (1) \iff (4) follows from Theorem 8.2. The last assertion on a bundle point follows from Theorem 8.1. \square

Example 11.4. Let us take the global parametrisation $\Phi := (s, t, t^2 + st^2, t^3)$ of the surface $X := \{(x, y, z, w) : z = y^2 + xy^2, y^2 = w\} \subset \mathbb{C}^4$. Then $\mathbb{C}[\Phi]^1$ coincides with Z in Example 7.7. All points of the form $(a, 0)$ ($a \neq -1$) are D -invariant of degree 1 but they are not bundle points of 1-jets of $\mathbb{C}[\Phi]^1$.

Bos and Calvi proves that D -invariance (gap-free property) is equivalent to the condition that the φ -Taylor projector is independent of the local parametrisation for plane algebraic curves ([BC₂], Theorem 3.4, 4.10). Unfortunately this can not be generalised. Taylorian property fails in the most simple 2-dimensional example as follows.

Example 11.5. Let us recall the surface $X \subset \mathbb{C}^3$ defined by $x_3 = x_2^2$ in Example 9.6. We have seen that the local parametrisation ψ is obtained by a linear transformation but the set of Bos-Calvi tangents of order 1 are different:

$$(D_{x,a}^{\varphi,1})^{\perp_X \perp_X} = D_{x,a}^{\varphi,1} \neq D_{x,a}^{\psi,1} = (D_{x,a}^{\psi,1})^{\perp_X \perp_X}.$$

Then the kernel $(D_{x,a}^{\varphi,1})^{\perp_X}$ and $(D_{x,a}^{\psi,1})^{\perp_X}$ of the Taylor projectors are different. Hence, no point of X are Taylorian although they are all bundle points.

Remark 11.6. For a general D -invariant point, we can only say that our Taylor projector of order d defines a structure of an Artinian algebra on $P^d(X_a)$. This structure comes from the structure of $\mathbb{C}[\Phi]^d = \mathbb{C}\{t\}/(\mathbb{C}[\Phi]_b^d)^{\perp_n}$ and it is unique up to canonical isomorphism by Theorem 8.1.

Remark 11.7. Concerning the relation among

- (1) bundle property of all the jet spaces of $\mathbb{C}[\Phi]_b^d$,
- (2) D -invariance of $(D_{x,a}^{\varphi,1})^{\perp_X}$ ($\mathbf{a} = \Phi(\mathbf{b})$),
- (3) Taylorian property at $\mathbf{a} = \Phi(\mathbf{b})$,

the author knows the examples above, $M_Z^{\text{bdl}} \subset M_Z^{\text{inv}}$ (Theorem 8.1) and 1-dimensional case treated in the next section.

12. TAYLORIAN PROPERTY OF POINTS ON EMBEDDED CURVES

If we restrict ourselves to the case of analytic curves the Taylor projector is well-defined at a general point, which generalise a theorem of Bos and Calvi [BC₂], 3.4 on plane algebraic curves.

Theorem 12.1. Let X be a 1-dimensional regular complex submanifold of a neighbourhood of $\mathbf{a} \in \mathbb{C}^m$. Then, for any $d \in \mathbb{N}$, the following three properties on $\mathbf{a} \in X$ are equivalent.

- (1) \mathbf{a} is a bundle point of all the jet spaces of $\mathbb{C}[\Phi]_b^d \downarrow$.
- (2) The powers of monomials appearing in $\mathbb{C}[\Phi]_b^d \downarrow \subset \mathbb{C}[\tau]$ form a gap-free sequence i.e. \mathbf{a} is D -invariant of degree d
- (3) \mathbf{a} is Taylorian of degree d .

Proof. For any local parametrisation Φ , the degree d part of $\mathbb{C}[\Phi]_t^d \downarrow$ is 0-dimensional or 1-dimensional. If \mathbf{a} satisfies the condition (1), they are constant in respective neighbourhoods of \mathbf{b} . If degree d part is 1-dimensional and degree $d + 1$ part is 0-dimensional on a neighbourhood V , the parts with higher degrees are all 0 on V . Then (2) holds. Since the vector space dimension of degree equal to or smaller than d part of $\mathbb{C}[\Phi]_t^d \downarrow$ is lower semi-continuous, if (2) holds, it holds in a neighbourhood also. This implies (1).

To prove (2) \implies (3), recall that every ideal of $\mathcal{O}_{X,a} \cong \mathbb{C}\{x\}$ is of the form $\mathfrak{m}_{X,a}^k$, a power of the maximal ideal. If \mathbf{a} is D -invariant of degree d , $\text{Ker } T_a^{\varphi,d} = (D_{X,a}^{\varphi,d})^{\perp_X}$ is an ideal and it is determined by $\dim_{\mathbb{C}} \mathcal{O}_{X,a} / (D_{X,a}^{\varphi,d})^{\perp_X} = \dim_{\mathbb{C}} P_{X,a}^d$. Hence it is independent of φ and we have $\text{Ker } T_a^{\varphi,d} = \text{Ker } T_a^{\psi,d}$ for any other local parametrisation ψ . Suppose that \mathbf{a} is D -invariant of degree d and take any $f \in \mathcal{O}_{X,a}$. Let us put $p := T_a^{\varphi,d}(f) \in P^d(X_a)$ and $q := T_a^{\psi,d}(f) \in P^d(X_a)$. In view of the retraction property of the projectors, we have

$$\begin{aligned} p - q &= T_a^{\varphi,d}(\kappa(p) - \kappa(q)) = T_a^{\varphi,d}((\kappa(p) - f) - (\kappa(q) - f)) \\ &\in T_a^{\varphi,d}(\text{Ker } T_a^{\varphi,d} - \text{Ker } T_a^{\psi,d}) = T_a^{\varphi,d}(\text{Ker } T_a^{\varphi,d} - \text{Ker } T_a^{\varphi,d}) = \{0\}, \end{aligned}$$

where $\kappa : P^d(X_a) \longrightarrow \mathcal{O}_{X,a}$ denotes the inclusion. This proves that $T_a^{\varphi,d}(f) = T_a^{\psi,d}(f)$. Thus D -invariance implies Taylorian property.

To prove the converse, we follow faithfully the idea of Bos-Calvi. The least space $\mathbb{C}[\Phi]_b^d \downarrow$ is a subspace of $\mathbb{C}[\tau]$. Suppose that \mathbf{a} is not D -invariant of degree d . Then there exists $s \in \mathbb{N}$ such that $\tau^s \notin \mathbb{C}[\Phi]_b^d \downarrow$ and $\tau^{s+1} \in \mathbb{C}[\Phi]_b^d \downarrow$. Let s be the maximum of such numbers. There exists a coordinate $x_j \in \mathbb{C}[x]$ such that $\varphi(x_j)_b \downarrow = \alpha \tau$ with $\alpha \neq 0$, otherwise the image of φ does not includes the elements of order 1, contradicting the retraction property. Let l denote the maximal number such that $\tau^{s+l} \in \mathbb{C}[\Phi]_b^d \downarrow$. Then $l \geq 1$ and

$$\tau^{s+1}, \tau^{s+2}, \dots, \tau^{s+l} \in \mathbb{C}[\Phi]_b^d \downarrow, \quad \tau^{s+l+1}, \tau^{s+l+2}, \dots \notin \mathbb{C}[\Phi]_b^d \downarrow$$

by the maximality of s . The least non-zero monomial appearing in $\varphi(x_j^s)$ is $\alpha^s t^s$. Taking $g_{s+i} \in \mathbb{C}[\Phi]^d$ such that $\varphi(g_{s+i})_b \downarrow = \tau^{s+i}$ ($i = 1, \dots, l$), we can eliminate the monomials of degree $s + 1, \dots, s + l$ appearing in $\varphi(x_j^s)$ by subtracting a linear combination $c_1 \varphi(g_{s+1}) + \dots + c_{s+l} \varphi(g_{s+l})$, beginning from g_{s+i} with smaller i . Then if we put

$$h := x_j^s - (c_1 g_{s+1} + \dots + c_{s+l} g_{s+l}),$$

we have

$$\varphi(h) = \alpha^s t^s + k(t) \cdot t^{s+l+1} \in (\mathbb{C}[\Phi]_b^d \downarrow)^{\perp_n}$$

for some $k(t) \in \mathcal{O}_n$. This proves that $T_a^{\varphi,d}(h) = 0$.

Now take another local parametrisation

$$\Psi = (\Phi_1(t' + t'^2), \dots, \Phi_n(t' + t'^2)).$$

The function $\psi(h)$ is expressed as

$$\psi(h) = \alpha^s (t' + t'^2)^s + k(t' + t'^2) \cdot (t' + t'^2)^{s+l+1}.$$

Here the coefficient of t'^{s+1} is not 0 and it follows that $S_n \langle \tau'^{s+1} | \psi(h) \rangle \neq 0$. Since $\tau^{s+1} \in \mathbb{C}[\Phi]_b^d \downarrow$ implies $\tau'^{s+1} \in \mathbb{C}[\Psi]_b^d \downarrow$ (see the proof of Theorem 11.1), we see that $T_a^{\psi,d}(h) \neq 0$. This is inconsistent with $T_a^{\varphi,d}(h) = 0$ and proves that \mathbf{a} is not a Taylorian point. \square

13. ZERO ESTIMATE AND TRANSCENDENCY INDEX

In this final section we see that the growth of the set of Bos-Calvi tangents of dual degree d measures the transcendence of the embedding of manifold germ X_a . In particular, we explain that the D -invariant property implies that the embedding of X_a has not a high index of transcendence.

First let us recall some known facts on zero-estimate on local algebras. The following invariant $\theta_\Phi(d)$ is called “ d -order” by Bos and Calvi in [BC₁]. It is more important than $\lambda_\Phi(d)$ ($\leq \theta_\Phi(d)$) defined in the previous section. This invariant coincides with one treated by the present author in [Iz₂], §1 and Example 6.2 as follows.

Definition 13.1. Let X be a submanifold of an open subset $\Omega \subset \mathbb{C}^m$ defined by ideal $I \subset \mathcal{O}_n(\Omega)$ and $\Phi := (\Phi_1, \dots, \Phi_m) : \mathbb{C}_b^n \rightarrow X_a$ a local parametrisation of the germ X_a . Let us use the abbreviation $f|_X := (f \bmod I)|_X$ (the restriction of $f \in \mathcal{O}_n(V)$ to $X \cap V$), $\mathbf{x}|_X := \{x_1|_X, \dots, x_m|_X\}$ and $A := \mathcal{O}_{X,a}$. The *zero-estimate function* is defined by

$$\begin{aligned} \theta_{A, \mathbf{x}|_X}(d) &= \theta_{\mathcal{O}_{n,b}, \Phi}(d) := \max \left\{ \deg p : p \in \mathbb{C}[\Phi]_b^d \downarrow \setminus \{0\} \right\} \\ &= \sup \{ \text{ord}_b F \circ \Phi : F \in \mathbb{C}[\mathbf{x}]^d, F \circ \Phi \neq 0 \} \\ &= \sup \{ \text{ord}_{X,a} F|_X : F \in \mathbb{C}[\mathbf{x}]^d, F|_X \neq 0 \} \end{aligned}$$

and the *transcendence index* of Φ by

$$\alpha(\Phi) := \limsup_{d \rightarrow \infty} \log_d \theta_{A, \mathbf{x}|_X}(d) = \limsup_{d \rightarrow \infty} \log_d \theta_{\mathcal{O}_{n,b}, \Phi}(d).$$

Note that the values of $\theta_{A, \mathbf{x}|_X}(d) = \theta_{\mathcal{O}_{n,b}, \Phi}(d)$ are all finite ([Iz₂], Proposition 1.1). The zero-estimate function $\theta_\Phi(d)$ and the transcendence index $\alpha(\Phi)$ are dependent on the embedding $X_a \subset \mathbb{C}_a^m$ but they are independent of the local parametrisation for a fixed $X_a \subset \mathbb{C}^m$.

Theorem 13.2 (Izumi [Iz₁]). *Let X be an n -dimensional regular complex submanifold ($n \geq 1$) of a neighbourhood of $\mathbf{a} \in \mathbb{C}^m$ and $\Phi : \mathbb{C}_b^n \rightarrow \mathbb{C}_a^m$ a local parametrisation of X at \mathbf{a} . Then the following conditions are equivalent.*

- (1) *The germ X_a is an analytic irreducible component of the germ of an algebraic set at \mathbf{a} .*
- (2) *There exists $a \geq 1$ and $b \geq 0$ such that*

$$\theta_{\mathcal{O}_{X,a}, \mathbf{x}|_X}(d) = \theta_{\mathcal{O}_{n,b}, \Phi}(d) \leq ad + b \quad \text{for all } d \in \mathbb{N}.$$

- (3) *There exists $a \geq 1$ and $b \geq 0$ such that*

$$\mathbb{C}[\Phi]^d \cap \mathfrak{m}_{n,b}^{ad+b+1} = \{0\} \quad \text{i.e.} \quad \mathbb{C}[\Phi]_b^d \downarrow \cap \frac{\mathfrak{m}_{n,b}^i}{\mathfrak{m}_{n,b}^{i+1}} = \{0\} \quad (i > ad + b).$$

- (4) $\alpha(\Phi) \leq 1$.

We use the name “transcendancy index” because of the equivalence $(1) \iff (4)$. Seemingly $\alpha(\Phi)$ is near to the excess number of algebraically independent generators of $\mathfrak{m}_{n,b}$ contained in Φ but it is known that it is not necessarily an integer in transcendence theory (cf. [P₁], [P₂], [P-W], see also a beginner’s note [Iz₂], Example 3.4). In [Iz₁] we treat polynomial functions on an analytically irreducible germ of analytic subset X of an open subset of \mathbb{C}^m and X need not be a smooth manifold. The equivalence of $(2) \iff (3)$ is trivial. A complete proof is given in [Iz₂], Theorem 2.3 in a stronger form.

The situation of this theorem may be well illustrated by the following.

Example 13.3 (Izumi [Iz₁]). Let C be the transcendental plane curve defined by $y = e^x - 1$. If we parametrise this by $\Phi = (t, e^t - 1)$, we have

$$\mathbb{C}[\Phi]^d = \text{Span} \left(\mathbb{C}[t]^d, \mathbb{C}[t]^{d-1} e^t, \mathbb{C}[t]^{d-2} e^{2t}, \dots, \mathbb{C}[t]^1 e^{(d-1)t}, e^{dt} \right).$$

This is just the space of solutions of the differential equation

$$D_t^{d+1} (D_t - 1)^d (D_t - 2)^{(d-1)} \cdots (D_t - d + 1)^2 (D_t - d)^1 f = 0 \quad (D_t := d/dt).$$

By the elementary theory of ordinary differential equations, for any $a \in \mathbb{C}$, there exists a unique solution f with

$$f^{(\nu)}(a) = \begin{cases} 0 & (0 \leq \nu \leq (d+1)(d+2)/2 - 2) \\ 1 & (\nu = (d+1)(d+2)/2 - 1) \end{cases}$$

and there exists no solution $f \neq 0$ with

$$f^{(\nu)}(a) = 0 \quad (0 \leq \nu \leq (d+1)(d+2)/2 - 1).$$

This proves that $\theta_{O_{1,a}, \{t-a, \exp(t-a)-1\}}(d) = (d+1)(d+2)/2 - 1$ and $\alpha = 2$ at all points $(a, e^a - 1) \in C$.

Remark 13.4. An example of a plane curve with an extremely transcendental point ($\alpha(\Phi) = \infty$) is given by Tetsuo Ueda (see [Iz₁], Example 2), using a gap power series, an analogue of Liouville constant.

Let \overline{X}_a denote the Zariski closure of X_a in \mathbb{C}^m , namely the smallest algebraic set that includes some representative of the germ X_a (germs are always taken with respect to the Euclidean topology). Then the Hilbert function of \overline{X}_a is defined by

$$\begin{aligned} \chi(\overline{X}_a, d) &:= \dim_{\mathbb{C}} P^d(X_a) - \dim_{\mathbb{C}} P^{d-1}(X_a) = \dim_{\mathbb{C}} D_{X,a}^{\varphi,d} - \dim_{\mathbb{C}} D_{X,a}^{\varphi,d-1} \\ &\quad \left(\dim_{\mathbb{C}} P^{-1}(X_a) = \dim_{\mathbb{C}} D_{X,a}^{\varphi,-1} = 0 \right) \end{aligned}$$

(cf. Definition 9.4). This is known to coincide with a polynomial of degree $\dim \overline{X}_a - 1$ in d for sufficiently large d .

Theorem 13.5. Let $\varphi : O_{m,a} \longrightarrow O_{n,b}$ ($n \geq 1$) be a local parametrisation of an embedded manifold germ $X_a \subset \mathbb{C}^m$ with component functions $\Phi = (\Phi_1, \dots, \Phi_m)$. Suppose that a is D -invariant of degree d , we have the zero-estimate inequalities:

$$\binom{n+d}{n} + \theta_{O_{n,b}, \Phi}(d) - d \leq \dim_{\mathbb{C}} \mathbb{C}[\Phi]^d = \sum_{i=0}^d \chi(\overline{X}_a, i) \leq \binom{m+d}{m}.$$

Hence, if a is D -invariant of degree ∞ , we have an estimate of the transcendancy index:

$$\alpha(\Phi) \leq \dim \overline{X}_a \leq m.$$

Proof. Note that

$$\dim_{\mathbb{C}} P^d(X_a) = \dim_{\mathbb{C}} \mathbb{C}[\Phi]^d$$

by the isomorphism stated in Definition 9.4. If we take $p \in \mathbb{C}[\Phi]_b^d$ with $\deg p = \theta_{A, \{\Phi_1, \dots, \Phi_m\}}(d)$ (the maximal degree), this dimension majorises the sum of the dimensions of the following linear subspaces:

- (1) the space $\bigoplus_{i=0}^d \mathfrak{m}_{n,b}^i / \mathfrak{m}_{n,b}^{i+1}$ appeared in Proposition 10.4;
- (2) the linear span of

$$\left\{ \frac{\partial^{|\nu|} p}{\partial \tau^\nu} : d < \text{ord}_{\mathfrak{m}} \frac{\partial^{|\nu|} p}{\partial \tau^\nu} < \infty, \nu \in \mathbb{N}_0^n \right\}$$

(by D -invariance).

Since the intersection of these spaces are $\{\mathbf{0}\}$, we have the left inequality of the first. The right inequality follows from

$$\dim_{\mathbb{C}} P^d(X_a) \leq \dim_{\mathbb{C}} \mathbb{C}[x]^d = \binom{m+d}{m}.$$

The first inequality in the theorem implies that

$$\theta_{O_{n,b}, \{\Phi_1, \dots, \Phi_m\}}(d) \leq \dim_{\mathbb{C}} \mathbb{C}[\Phi]^d = \dim_{\mathbb{C}} P^d(X_a) = \sum_{i=0}^d \chi(\bar{X}_a, i).$$

Since the Hilbert function of \bar{X}_a coincide with a polynomial of degree $\dim \bar{X}_a - 1$ for sufficiently large d , the last term is comparable to $d^{\dim \bar{X}_a}$ and the inequality on $\alpha(\Phi)$ follows. \square

Let $\Phi : U \longrightarrow \mathbb{C}^m$ be an embedding of an open subset of \mathbb{C}^n onto a submanifold $X \subset \mathbb{C}^m$ i.e. Φ induces a biholomorphic homeomorphism onto the image. We have seen that the complement of the set of bundle points of all the jet spaces of $\mathbb{C}[\Phi]^d$ for all $d \in \mathbb{N}$ is contained in a countable union of thin closed analytic subsets. Then the zero-estimate inequality in Theorem 13.5 implies the following global result.

Corollary 13.6. *Let X be an n -dimensional regular complex submanifold of an open subset $\Omega \subset \mathbb{C}^m$ ($n \geq 1$). Then there exists a countable union A of thin closed analytic subsets of X such that, for any local parametrisation Φ at $a \in \Omega \setminus A$, we have $\alpha(\Phi) \leq \dim \bar{X}_a \leq m$. Hence the set A is of first category in Baire's sense and with Lebesgue measure 0 in X .*

Remark 13.7. Gabrielov gives a zero-estimate [Ga], Theorem 5 of Noetherian functions on an integral curve of a Noetherian vector field (see [GK] also). It yields a zero-estimate of Noetherian functions on \mathbb{C}^n immediately as follows. Suppose that

$$\Psi := \{x, \Phi\} \subset O_{n,b}$$

is a join of an affine coordinate system $x := (x_1, \dots, x_n)$ and a Noetherian chain $\Phi := \{\Phi_1, \dots, \Phi_m\}$ of order m , which means that

$$\frac{\partial \Phi_i}{\partial x_j} = P_{ij}(x_1, \dots, x_n, \Phi_1, \dots, \Phi_m) \quad (i = 1, \dots, m; j = 1, \dots, n)$$

for some polynomials P_{ij} . This Ψ is the set of the mapping components of the embedding onto the graph $X \subset \mathbb{C}^{m+n}$ of the Noetherian chain $\{\Phi_1, \dots, \Phi_m\}$. Then we have

$$\alpha(\Psi) \leq 2(m+n)$$

(cf. [Iz₄], Corollary 12).

Our Corollary 13.6 gives a slightly stronger estimate

$$\alpha(\Psi) \leq \dim \bar{X}_a \leq m+n$$

for (only for) a complement of a small subset of X without the Noetherian condition. In view of Remark 13.4, exclusion of some point set is inevitable for our general analytic case.

Proposition 13.8. *Let $X_a \subset \mathbb{C}^m$ and $X'_a \subset \mathbb{C}^m$ be affine equivalent germs of embedded manifolds, i.e. there exists an affine transformation $\Theta : \mathbb{C}^m \rightarrow \mathbb{C}^m$ which maps $X_a \subset \mathbb{C}^m$ to $X'_a \subset \mathbb{C}^m$ biholomorphically. Then $X_a \subset \mathbb{C}^m$ and $X'_a \subset \mathbb{C}^m$ coincide to have the properties of bundle point, D -invariance, Taylorian and they have the same invariant $\theta(d)$ and α .*

Proof. Let Φ be a local parametrisation of X_a . Then $\Phi' := \Theta \circ \Phi$ is a local parametrisation of X'_a . Since Θ is affine, $\mathbb{C}[\Phi]^d = \mathbb{C}[\Phi']^d$ and hence $\mathbb{C}[\Phi]_b^d \downarrow = \mathbb{C}[\Phi']_b^d \downarrow$. This implies every thing. \square

It is easy to see that all germs at the points on the curve $y = x^2$ are mutually affine equivalent. In the case of $y = x^n$ ($n \in \mathbb{N}$, $n \geq 2$), the germs at points except $(0, 0)$ are all affine equivalent. Of course $\alpha = 1$ in these cases because they are algebraic. It is interesting that the germs at points on the transcendental curve $y = \exp x - 1$ are also affine equivalent. In this case $\theta(d)$ and α are already given in Example 13.3.

Acknowledgements The author thanks Tohru Morimoto for invaluable suggestions. The author is also grateful to Takashi Aoki, Yutaka Matsui and Yayoi Nakamura for helpful discussions.

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